PRÜFER $\star$–MULTIPLICATION DOMAINS AND $\star$–COHERENCE

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1. Introduction and Background

The purpose of this paper is to deepen the study of the Prüfer $\star$–multiplication domains, where $\star$ is a semistar operation (the definitions are recalled later in this section). For this reason, in Section 2, we introduce the $\star$–domains, as a natural extension of the v–domains [19, page 418], where v is the classical Artin's divisorial operation. We investigate their close relation with the Prüfer $\star$–multiplication domains. In particular, in Section 3, we obtain a characterization of Prüfer $\star$–multiplication domains in terms of $\star$–domains satisfying a variety of coherent-like conditions. In Section 4, we extend to the semistar setting the notion of $\mathtt{H}$–domain introduced by Glaz and Vasconcelos [22, Remark 2.2 (c)] and we show, among the other results that, in the class of the $\mathtt{H}(\star)$–domains, the Prüfer $\star$–multiplication domains coincide with the $\star$–domains.

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D–submodules of K and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e. $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D–submodules of K. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

A *semistar operation* on D is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D), E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(xE)^\star = xE^\star$;
$(\star_2)$ $E \subseteq F$ implies $E^\star \subseteq F^\star$;
$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Recall that, given a semistar operation $\star$ on D, for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following basic formulas follow easily from the axioms:

$$
\begin{array}{c}
(EF)^\star = (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star \ ; \\
(E+F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star \ ; \\
(E:F)^\star \subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star \,, \ \text{if} \ (E:F) \neq 0; \\
(E \cap F)^\star \subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star \,, \ \text{if} \ E \cap F \neq (0) \,;
\end{array}
$$

cf. for instance [11, Theorem 1.2 and p. 174].

A *(semi)star operation* is a semistar operation that, restricted to $\boldsymbol{F}(D)$, is a star operation (in the sense of [19, Section 32]). It is easy to see that a semistar operation $\star$ on D is a (semi)star operation if and only if $D^\star = D$.

If $\star$ is a semistar operation on D, then we can consider a map $\star_{_f} : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined, for each $E \in \overline{\boldsymbol{F}}(D)$, as follows:

$$E^{\star_f} := \bigcup\{F^\star \mid \ F \in \boldsymbol{f}(D) \text{ and } F \subseteq E\}.$$

It is easy to see that $\star_{_f}$ is a semistar operation on D, called *the semistar operation of finite type associated to* $\star$. Note that, for each $F \in \boldsymbol{f}(D)$, $F^\star = F^{\star_f}$. A semistar

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operation $\star$ is called a *semistar operation of finite type* if $\star = \star_f$. It is easy to see that $(\star_f)_f = \star_f$ (that is, $\star_f$ is of finite type).

If T is an overring of D, we can define a semistar operation on D, denoted by $\star_{\{T\}}$ and defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\boldsymbol{F}}(D)$. It is easily seen that $\star_{\{T\}}$ is a semistar (non (semi)star, if $D \subsetneq T$) operation of finite type.

If $\star_1$ and $\star_2$ are two semistar operations on D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$. This is equivalent to say that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$, for each $E \in \overline{\boldsymbol{F}}(D)$. Obviously, for each semistar operation $\star$, we have $\star_f \leq \star$.

We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* if $I^\star \cap D = I$, a *quasi-$\star$-prime* if it is a prime quasi-$\star$-ideal, and a *quasi-$\star$-maximal* if it is maximal in the set of all quasi-$\star$-ideals. A quasi-$\star$-maximal ideal is a prime ideal. It is possible to prove that each quasi-$\star_f$-ideal is contained in a quasi-$\star_f$-maximal ideal. More details can be found in [15, page 4781]. We will denote by $\mathcal{M}(\star_f)$ the set of the quasi-$\star_f$-maximal ideals of D.

If Δ is a set of prime ideals of an integral domain D, then the semistar operation $\star_\Delta$ defined on D as follows

$$E^{\star_\Delta} := \bigcap\{ED_P \mid P \in \Delta\}\,, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D)\,,$$

is called *the spectral semistar operation associated to* Δ. A semistar operation $\star$ of an integral domain D is called *a spectral semistar operation* if there exists a subset Δ of the prime spectrum of D, $\mathrm{Spec}(D)$, such that $\star = \star_\Delta$.

When $\Delta := \mathcal{M}(\star_f)$, we set $\widetilde{\star} := \star_{\mathcal{M}(\star_f)}$, i.e.

$$E^{\widetilde{\star}} := \bigcap\left\{ED_M \mid M \in \mathcal{M}(\star_f)\right\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D).$$

A semistar operation $\star$ is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$, for each $E, F \in \overline{\boldsymbol{F}}(D)$. Spectral semistar operations are stable [11, Lemma 4.1 (3)].

We recall from [12, Chapter V] that *a localizing system of ideals of* D is a family $\mathcal{F}$ of ideals of D such that:

- **(LS1)** If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$.
- **(LS2)** If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$, for each $i \in I$, then $J \in \mathcal{F}$.

A localizing system $\mathcal{F}$ is *finitely generated* if, for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$.

The relation between stable semistar operations and localizing systems has been deeply investigated by M. Fontana and J. Huckaba in [11] and by F. Halter-Koch in the context of module systems [24]. We summarize some of results that we need in the following Proposition (see [11, Proposition 2.8, Proposition 3.2, Proposition 2.4, Corollary 2.11, Theorem 2.10 (B)]).

Proposition 1. *Let D be an integral domain.*

- **(1)** *If $\star$ is a semistar operation on D, then $\mathcal{F}^\star := \{I \text{ ideal of } D \mid I^\star = D^\star\}$ is a localizing system (called* the localizing system associated to $\star$*).*
- **(2)** *If $\star$ is a semistar operation of finite type, then $\mathcal{F}^\star$ is a finitely generated localizing system.*
- **(3)** *Let $\star_{\mathcal{F}}$ or, simply, $\overline{\star}$ be the semistar operation associated to a given localizing system $\mathcal{F}$ of D and defined by $E \mapsto E^{\overline{\star}} := \bigcup\{(E : J) \mid J \in \mathcal{F}\}$, for each $E \in \overline{\boldsymbol{F}}(D)$. Then $\star_{\mathcal{F}}$ (called* the semistar operation associated to the localizing system $\mathcal{F}$*) is a stable semistar operation on D.*
- **(4)** *$\overline{\star} \leq \star$ and $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}}$.*
- **(5)** *$\overline{\star} = \star$ if and only if $\star$ is stable.*
- **(6)** *If $\mathcal{F}$ is a finitely generated localizing system, then $\star_{\mathcal{F}}$ is a finite type (stable) semistar operation.*

(7) $\mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f := \{I \in \mathcal{F}^\star \mid I \supseteq J,$ *for some finitely generated ideal* $J \in \mathcal{F}^\star\}$ *and* $\widetilde{\star} = \overline{\star_f}$, *i.e.* $\widetilde{\star}$ *is the stable semistar operation of finite type associated to the localizing system* $\mathcal{F}^{\star_f}$.

(8) *If* $\mathcal{F}'$ *and* $\mathcal{F}''$ *are two localizing systems of* D, *then* $\mathcal{F}' \subseteq \mathcal{F}''$ *if and only if* $\star_{\mathcal{F}'} \leq \star_{\mathcal{F}''}$. □

By v_D (or, simply, by v) we denote the v–(semi)star operation defined as usual by $E^v := (D : (D : E))$, for each $E \in \overline{\boldsymbol{F}}(D)$. By t_D (or, simply, by t) we denote $(v_D)_f$ the t–(semi)star operation on D and by w_D (or just by w) the stable semistar operation of finite type associated to v_D (or, equivalently, to t_D), considered by Wang Fanggui and R.L. McCasland in [36]; i.e. $w_D := \widetilde{v_D} = \widetilde{t_D}$.

If $I \in \overline{\boldsymbol{F}}(D)$, we say that I is *$\star$–finite* if there exists $J \in \boldsymbol{f}(D)$ such that $J^\star = I^\star$. It is immediate to see that if $\star_1 \leq \star_2$ are semistar operations and I is $\star_1$–finite, then I is $\star_2$–finite. In particular, if I is $\star_f$–finite, then it is $\star$–finite. The converse is not true and it is possible to prove that I is $\star_f$–finite if and only if there exists $J \in \boldsymbol{f}(D)$, $J \subseteq I$, such that $J^\star = I^\star$ [17, Lemma 2.3].

If I is a nonzero ideal of D, we say that I is *$\star$–invertible* if $(II^{-1})^\star = D^\star$, i.e., if $II^{-1} \in \mathcal{F}^\star$. We denote by $\mathrm{Inv}(D, \star)$ the set of all the $\star$–invertible ideals of D. From the definitions, it follows easily that an ideal is $\star$–invertible if and only if it is $\overline{\star}$–invertible (and so I is $\widetilde{\star}$–invertible if and only if I is $\star_f$–invertible). If I is $\star_f$–invertible, then I and I^{-1} are $\star_f$–finite [17, Proposition 2.6].

A domain D is called *a Prüfer $\star$–multiplication domain* (for short, *P$\star$MD*) if each nonzero finitely generated ideal is $\star_f$-invertible (cf. for instance [27] and [13]). When $\star = v$ we have the classical notion of PvMD (cf. for instance [23], [33] and [30]); when $\star = d$, where d denotes *the identity (semi)star operation,* we have the notion of Prüfer domain [19, Theorem 22.1].

We say that a semistar operation $\star$ on D is *a.b.* (= arithmetisch brauchbar) if, for each $E \in \boldsymbol{f}(D)$ and for all $F, G \in \overline{\boldsymbol{F}}(D)$, $(EF)^\star \subseteq (EG)^\star$ implies $F^\star \subseteq G^\star$ and $\star$ is *e.a.b.* (= endlich arithmetisch brauchbar) if the same holds for all $E, F, G \in \boldsymbol{f}(D)$. Obviously, a.b. implies e.a.b.; in case of semistar operations of finite type, it is easy to see that the notions of e.a.b. and a.b. semistar operation coincide (in this situation, we will write (e.)a.b. operation).

Finally, let $\star_1 \leq \star_2$ be two semistar operations on D, then we say that D is *a $(\star_1, \star_2)$–domain* if $\star_1 = \star_2$.

2. Prüfer $\star$–multiplication domains and $\star$–domains

Let D be a domain and $\star$ a semistar operation on D. We say that D is *a $\star$–domain* if $(II^{-1})^\star = D^\star$ for each $I \in \boldsymbol{f}(D)$.

For $\star = v$ we have the notion of v–domain considered in [19, Section 34], and for $\star = d$ we have that the notions of d–domain, PdMD and Prüfer domain coincide [19, Theorem 22.1].

Proposition 2. *Let D be a domain and let $\star_1, \star_2$ be two semistar operations on D.*

(1) *If $\star_1 \leq \star_2$ and D is a $\star_1$–domain then D is a $\star_2$–domain.*
(2) *The following statements are equivalent:*
 (i) *D is a $\star_f$–domain;*
 (ii) *D is a $\widetilde{\star}$–domain;*
 (iii) *D is a P$\star$MD.*
(3) *A P$\star$MD is always a $\star$–domain.*
(4) *The following statements are equivalent:*

(j) *D is a $\star$–domain;*
(jj) *D is a $\overline{\star}$–domain.*

(5) *Let $v(D^\star)$ be the semistar operation on D defined by $E \mapsto E^{v(D^\star)} := (D^\star : (D^\star : E))$, for each $E \in \overline{\boldsymbol{F}}(D)$ (cf.* [17, Lemma 2.11 (4) and its proof] *or* [34, Example 1.8 (2)]*). If D is a $\star$–domain, then D is a $v(D^\star)$–domain and $\star_{_f} = (v(D^\star))_{_f}$.*

Proof. (1) follows immediately from the definitions.

(2) follows from [17, Proposition 2.18] and from the definition of a P$\star$MD. (Note that a P$\star$MD coincides with a P$\widetilde{\star}$MD and with a P$\star_{_f}$MD, cf. also [13, page 30].)

(3) is an easy consequence of (1) and (2), since $\star_{_f} \leq \star$.

(4) We have already observed that $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}}$, thus $II^{-1} \in \mathcal{F}^\star$ if and only if $II^{-1} \in \mathcal{F}^{\overline{\star}}$.

(5) Since $\star \leq v(D^\star)$, for each semistar operation $\star$ [34, Corollary 3.8], the first assertion is an immediate consequence of (1). The second assertion follows by [17, Remark 2.13 (c)]. $\square$

It is known that, when $\star$ is a star operation, a P$\star$MD is a PvMD such that $\star_{_f} = t$ [13, Proposition 3.4]. The next result extends the previous characterization to the case of $\star$–domains.

Corollary 1. *Let $\star$ be a star operation defined on an integral domain D. If D is a $\star$–domain then D is a v–domain and $\star_{_f} = t$.*

Proof. Since in the present situation $D^\star = D$, the statement is a straightforward consequence of Proposition 2 (5). $\square$

Remark 1. (1) As a consequence of the previous result we re-obtain that the notions of PvMD, PtMD and PwMD coincide (cf. [2, Theorem 2.18] and [13, Remark 3.1 and Corollary 3.1]).

(2) Note that $\star$–domains are not always P$\star$MD, even if $\star$ is a (semi)star operation. For instance, recall that an essential domain is a v–domain [19, Proposition 44.13] and not every essential domain is a PvMD [26] (cf. also [25] for an example of an essential domain with a non-essential localization, and so, in particular, which is not PvMD [33, Example 2.1, Proposition 1.1 and Corollary 1.4]). An example of a star operation $\star$, defined on an essential domain D, such that D is a $\star$–domain but not a P$\star$MD is given in the following Example 2.

(3) Note also that, from Propostion 2 (2) and the previous observation (2), we deduce in particular that the notions of t–domain and w–domain coincide, but they are strictly stronger than the notion of v–domain (as observed in (2)).

(4) Note that from Proposition 2 (1, 2), we deduce that if $\star_1 \leq \star_2$ and D is a P$\star_1$MD then D is also a P$\star_2$MD. Since $\widetilde{\star} \leq \overline{\star} \leq \star$, we have that a P$\widetilde{\star}$MD is a P$\overline{\star}$MD, which is a P$\star$MD and thus it is easy to see that all these notions coincide (cf. Proposition 2 (2)).

(5) In [19, Section 34], a v–domain is defined as a domain such that the v–operation is e.a.b., and in [19, Theorem 34.6] it is shown that this is equivalent to the fact that each finitely generated ideal is v–invertible. This type of characterization does not hold for general semistar operations $\star$ (cf. the following Example 1).

Proposition 3. *Let D be an integral domain and $\star$ a semistar operation on D. If D is a $\star$–domain (e.g. a P$\star$MD) then $\star$ is an a.b. operation. In particular, a $\star$–domain D is quasi–$\star$–integrally closed (i.e. $D^\star = \bigcup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\}$) and so $D^\star$ is integrally closed.*

Proof. If $(FG)^\star \subseteq (FH)^\star$ with $F \in \boldsymbol{f}(D)$ and $G, H \in \overline{\boldsymbol{F}}(D)$, then $G^\star = ((FF^{-1})^\star G)^\star = (FF^{-1}G)^\star \subseteq (FF^{-1}H)^\star = ((FF^{-1})^\star H)^\star = H^\star$. The other statements follow respectively by [8, Lemma 4.13] and by [14, Proposition 4.3]. □

Note that it is not always true that if $\star$ is a.b. then D is a $\star$–domain, as the following examples show.

Example 1. **(1)** *An a.b. non-stable semistar (non-star) operation of finite type $\star$ on an integral domain D such that D is not a $\star$–domain (or, equivalently, not a P$\star$MD).*

Take a pseudo-valuation non-valuation domain D, with maximal ideal M, and set $V := M^{-1}$. Let $\star = \star_f := \star_{\{V\}}$. It is easy to see that $\star$ is an a.b. semistar operation on D. Since $\mathcal{M}(\star_f) = \{M\}$, then D is not a P$\star$MD [27, Theorem 1.1 ((1)⇔(4))] and so it is not a $\star$–domain, because in this case $\star$ is of finite type, so D is a $\star$-domain if and only if it is a P$\star$MD (cf. Proposition 2 (2))). Finally $\star$ is not stable, because otherwise $\star = \widetilde{\star}$ [11, Corollary 3.9 (2)] and hence the fact that $\widetilde{\star}$ is (e.)a.b. implies that D is a P$\star$MD [13, Theorem 3.1].

(2) *An a.b. non-stable star operation of finite type $\star$ on an integral domain D such that D is not a $\star$–domain.*

It is easy to check that the v–operation is e.a.b. if and only if the t–operation is a.b. (cf. also [16, Definition 3.6 and Lemma 3.9 (2)]). Therefore, a v–domain is an integral domain such that the t–operation is a.b. (Remark 1 (5)), but we already observed (Remark 1 (2)) that a v–domain is not necessarily a t–domain (that is, a PvMD). The non-stability of the operation t, on a v–domain which is not a PvMD, follows from the same argument as in the previous example.

Example 2. *An (a.b.) star operation $\star$ on an integral domain D such that D is a $\star$–domain but not a P$\star$MD.*

Let D be a domain and let $\{V_\alpha\}$ be a nonempty set of valuation overrings of D which are essential for D (that is, V_α is the localization of D at its center P_α in D). Consider the semistar operation $\star$ induced by this set overrings (i.e. $E^\star := \bigcap_\alpha EV_\alpha$, for each $E \in \overline{\boldsymbol{F}}(D)$; thus $\star$ is a semistar non-(semi)star operation on D if $D \subsetneq \bigcap_\alpha V_\alpha$). Let I be a nonzero finitely generated ideal of D, then $(II^{-1})^\star = \bigcap_\alpha (I(D:I))V_\alpha = \bigcap_\alpha I(D:I)D_{P_\alpha} = \bigcap_\alpha (ID_{P_\alpha}(D_{P_\alpha} : ID_{P_\alpha})) = \bigcap_\alpha IV_\alpha(V_\alpha : IV_\alpha) = \bigcap_\alpha V_\alpha = D^\star$. Thus, each nonzero finitely generated ideal of D is $\star$–invertible and so D is a $\star$–domain.

Note that a similar argument shows that $\star$ is stable: $(E\cap F)^\star = \bigcap_\alpha (E\cap F)D_{P_\alpha} = \bigcap_\alpha (ED_{P_\alpha} \cap FD_{P_\alpha}) = (\bigcap_\alpha ED_{P_\alpha}) \cap (\bigcap_\alpha FD_{P_\alpha}) = E^\star \cap F^\star$, for all $E, F \in \overline{\boldsymbol{F}}(D)$ (i.e. $\star = \overline{\star}$).

Note also that a semistar operation defined by a family of valuation overrings (like the $\star$ defined above) is necessarily a.b..

Assume from now that $D = \bigcap_\alpha V_\alpha$. Note that, in this case, $\star$, defined on $\boldsymbol{F}(D)$, is a star operation on D, thus $\star \leq v$ and so D is also a v–domain. By Proposition 2(5), we can deduce that $\star_f = t$. So, D is a P$\star$MD (= P$\star_f$MD) if and only if it is a PvMD (= PtMD). We can conclude that if you choose D not to be a PvMD (such example exists (Remark 1 (2)), then D is a $\star$-domain (and a v–domain) which is not a P$\star$MD (nor a a PvMD).

In this situation, $\star$ may be not of finite type, since if $\star = \star_f$, then $\star = \overline{\star} = \widetilde{\star}$, thus $\widetilde{\star}$ would be a.b. and so D would be a P$\star$MD [13, Theorem 3.1 ((v)⇒(i))]. Finally, note that if $\overline{v} \neq v$, then necessarily $\star \lneq v$.

Proposition 4. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(i) *D is a $\star$–domain [respectively: a P$\star$MD].*

(ii) *for all $E, F \in \boldsymbol{f}(D)$ there exists $H \subseteq (E : F)$, $H \in \boldsymbol{F}(D)$ [respectively: $H \in \boldsymbol{f}(D)$], such that $E^\star = (FH)^\star$.*

(iii) *for all $E, F \in \boldsymbol{f}(D)$, $(F(E : F))^\star = E^\star$ [respectively: $(F(E : F))^{\star_f} = E^\star$].*

Proof. (i)⇒(ii) Assume that D is a $\star$–domain and take $H := F^{-1}E$. Clearly $HF = FF^{-1}E \subseteq DE = E$ and so $H \subseteq (E : F)$. Moreover, $(FH)^\star = (FF^{-1}E)^\star = ((FF^{-1})^\star E)^\star = E^\star$. If D is a P$\star$MD, let $G \in \boldsymbol{f}(D)$ such that $G \subseteq F^{-1}$ and $G^\star = (F^{-1})^\star$. In this case, we just need to modify the choice of H, setting $H := GE$.

(ii)⇒(iii) is straightforward, since $H \subseteq (E : F)$ [and, in the parenthetical statement, $H \in \boldsymbol{f}(D)$].

(ii)⇒(iii) is obvious by taking $E = D$. □

Remark 2. Note that the proof of Proposition 4 shows that D *is a $\star$–domain [respectively: P$\star$MD] if and only if* $(F(E : F))^\star = E^\star$ *[respectively: $(F(E : F))^{\star_f} = E^{\star_f}$], for all $F \in \boldsymbol{f}(D)$ and $E \in \overline{\boldsymbol{F}}(D)$.*

We are in condition to give a characterization of the $\star$–domains [respectively: Prüfer $\star$–multiplication domains] by using that $\star$ is a.b. or that D is quasi–$\star$–integrally closed (Proposition 3).

Corollary 2. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(i) *D is a $\star$–domain [respectively: a P$\star$MD].*

(ii) *$\star$ is a.b. and $(EF^{-1})^\star = (E^\star : F)$ [respectively: $(EF^{-1})^{\star_f} = (E^{\star_f} : F)$] for all $F \in \boldsymbol{f}(D)$ and $E \in \overline{\boldsymbol{F}}(D)$.*

(iii) *D is a quasi–$\star$–integrally closed domain and $(EF^{-1})^\star = (E^\star : F)$ [respectively: $(EF^{-1})^{\star_f} = (E^{\star_f} : F)$] for all $F \in \boldsymbol{f}(D)$ and $E \in \overline{\boldsymbol{F}}(D)$.*

Proof. We show the equivalences for the $\star$–domain case; the equivalences among the parenthetical statements follow from the fact that a P$\star$MD coincide with a $\star_f$–domain (Proposition 2 (2)).

(i)⇒(ii)⇒(iii). By Proposition 3 and by the fact that $E^\star = (FF^{-1}E)^\star \subseteq (F(E : F))^\star \subseteq (F(E^\star : F))^\star \subseteq E^\star$ we have $(FF^{-1}E)^\star = (F(E^\star : F))^\star$. Since $\star$ is a.b. we deduce that $(F^{-1}E)^\star = (E^\star : F)^\star = (E^\star : F)$.

(iii)⇒(i) By taking $E = F$ we have that $(FF^{-1})^\star = (F^\star : F)$. Moreover, by the fact that D is quasi–$\star$–integrally closed, we have $D^\star \subseteq (F^\star : F) = (F^\star : F^\star) \subseteq \bigcup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\} = D^\star$, and so $(F^\star : F) = D^\star$. □

Note that the previous corollary generalizes to the semistar setting some characterizations of the v–domains proved in [1, Theorem 2].

The next goal is to relate $\star$–domains with properties of stability for the semistar operation $\star$.

Proposition 5. *Let D be a $\star$–domain. Assume that D is integrally closed. Then, for all $E, F \in \boldsymbol{f}(D)$:*

$$(E :_D F)^\star = (E^\star :_{D^\star} F)\,.$$

In particular, D is a $((\overline{\star})_f, \star_f)$–domain and so $(E \cap F)^\star = E^\star \cap F^\star$.

Proof. If D is integrally closed, then for each $E \in \boldsymbol{f}(D)$, $(E : E) = D$ [19, Proposition 34.7]. Note that $(E :_D F) = (E : F) \cap D = (E : F) \cap (E : E) = (E : F + E)$. On the other hand, we know already that in a $\star$–domain, $(E^\star : E^\star) = D^\star$ and $(EF^{-1})^\star = (E^\star : F)$ (Proposition 3 and Corollary 2). Since, in general, $(EF^{-1})^\star \subseteq (E : F)^\star \subseteq (E^\star : F)$, we deduce that $(E : F)^\star = (E^\star : F)$. Moreover $(E : F + E)^\star = (E^\star : F + E) = (E^\star : (F + E)^\star) = (E^\star : (F^\star + E^\star)^\star) = (E^\star : (F^\star + E^\star)) = (E^\star : F^\star) \cap (E^\star : E^\star) = (E^\star : F^\star) \cap D^\star = (E^\star :_{D^\star} F^\star) = (E^\star :_{D^\star} F)$.

In order to prove the second part of the statement we show that, for each $E \in \boldsymbol{f}(D)$, $E^\star = E^{\overline{\star}}$.

$$\begin{array}{rl} x \in E^\star \Leftrightarrow & 1 \in (E^\star : xD) \Leftrightarrow (E^\star :_{D^\star} xD) = D^\star \Leftrightarrow (E :_D xD)^\star = D^\star \\ \Leftrightarrow & (E :_D xD)^\star \in \mathcal{F}^\star \Leftrightarrow I \subseteq (E : xD) \quad \text{for some } I \in \mathcal{F}^\star \\ \Leftrightarrow & xI \subseteq E \quad \text{for some } I \in \mathcal{F}^\star \Leftrightarrow x \in E_{\mathcal{F}^\star} = E^{\overline{\star}}. \end{array}$$

Finally, since $\overline{\star}$ is stable and $(\overline{\star})_{_f} = \star_{_f}$, then $E^\star \cap F^\star = E^{\overline{\star}} \cap F^{\overline{\star}} = (E \cap F)^{\overline{\star}} \subseteq (E \cap F)^\star \subseteq E^\star \cap F^\star$, when $E, F \in \boldsymbol{f}(D)$). □

Remark 3. **(1)** Note that, if $\star$ is a (semi)star operation on D and D is a $\star$–domain, then D is integrally closed, thus the previous Proposition 5 generalizes to the semistar setting a result proved recently by Anderson and Clarke [3, Theorem 2.8].

(2) In relation with Proposition 5, we remark that it is possible to generalize in the semistar setting a result proved by Anderson and Cook [2, Theorem 2.6]. More precisely, *if $\star$ is a semistar operation on an integral domain, then the following conditions are equivalent:*

- **(i)** *D is a $((\overline{\star})_{_f}, \star_{_f})$–domain [respectively: $(\overline{\star}, \star)$–domain].*
- **(ii)** *For all $E, F \in \boldsymbol{f}(D)$ [respectively: $E, F \in \overline{\boldsymbol{F}}(D)$], $(E \cap F)^\star = E^\star \cap F^\star$.*
- **(iii)** *For all $E, F \in \boldsymbol{f}(D)$ [respectively: $E \in \overline{\boldsymbol{F}}(D)$, $F \in \boldsymbol{f}(D)$], $(E :_D F)^\star = (E^\star :_{D^\star} F)$.*
- **(iv)** *For each $E \in \boldsymbol{f}(D)$ [respectively: $E \in \overline{\boldsymbol{F}}(D)$], and for each nonzero element $x \in K$, $(E :_D xD)^\star = (E^\star :_{D^\star} xD)$.*

Clearly, if $\star$ is stable (i.e. $\overline{\star} = \star$), then all the previous statements hold.

The implications (iii)⇒(iv)⇒(i)⇒(ii) are essentially proved in Proposition 5.

(ii)⇒(iii). If $F = x_1D + x_2D + \cdots + x_nD$, then $(E :_D F)^\star = (E :_D (x_1D + x_2D + \cdots + x_nD))^\star = (\bigcap\{E :_D x_iD) \mid 1 \leq i \leq n\})^\star = (\bigcap\{x_i^{-1}E \cap D \mid 1 \leq i \leq n\})^\star = \bigcap\{x_i^{-1}E^\star \cap D^\star \mid 1 \leq i \leq n\} = \bigcap\{E^\star :_{D^\star} x_iD) \mid 1 \leq i \leq n\} = (E^\star :_{D^\star} F)$.

(3) It is easily seen that, *if $\star$ is a semistar operation of finite type on D, D is a $((\overline{\star})_{_f}, \star_{_f})$–domain if and only if $\star$ is a stable semistar operation.* Thus a semistar operation of finite type is stable if and only if the (non-parenthetical) equivalent conditions in (2) are satisfied.

(4) As already observed in the star setting, if D is an integrally closed $((\overline{\star})_{_f}, \star_{_f})$–domain then D is not necessarily a $\star$–domain. (Take D to be an integrally closed non-Prüfer domain and $\star = d$.)

However, in the particular case that $\star = v$, then *D is a v–domain if and only if D is integrally closed $((\overline{v})_{_f}, t)$–domain.* (The "if part" is due to Anderson et al. [1, Theorem 7], cf. also (1), (2) and Proposition 5; the "only if " part was proved by Matsuda and Okabe [31], cf. also [3, Theorem 2.8].)

At this point, for the general case, if we replace the condition "D is integrally closed" with the condition "$\star$ is a.b. on D" (which is a stronger condition in the (semi)star setting), it is natural to ask:

(Q-1) *Let $\star$ be a semistar operation on D. Is it true that D is a $\star$–domain if and only if $\star$ is a.b. and D is a $((\overline{\star})_{_f}, \star_{_f})$–domain ?*

Note that the answer to the previous question is positive for $\star$ of finite type, since in this case $(\overline{\star})_{_f} = \widetilde{\star}$ and we know that D is a P$\star$MD if and only if $\widetilde{\star} = \star_{_f}$ is (e.)a.b. [13, Theorem 3.1], cf. also the following Theorem 3.

There is another important case in which the answer to **(Q-1)** is positive. Let $\star := \star_\Delta$ be a spectral semistar operation, where $\Delta \subseteq \mathrm{Spec}(D)$. Clearly $\star$ is stable and so D is a $((\overline{\star})_{_f}, \star_{_f})$–domain.

Assume that $\star$ is a.b.. For each $P \in \Delta$, let $\iota_{_P}$ be the canonical embedding of D in D_P. We claim that $\star_{\iota_P}$ coincides with d_{D_P} (i.e. the identity (semi)star

operation of D_P), for each $P \in \Delta$. In fact, if $E \in \overline{\boldsymbol{F}}(D_P)$, $E \subseteq E^{\star_{\iota_P}} = E^\star = \bigcap_{P_\alpha \in \Delta} ED_{P_\alpha} \subseteq ED_P = E$. Moreover, $\star_{\iota_P}$ $(= d_{D_P})$ is also a.b. by [34, Proposition 3.1 (4)]. Thus, each finitely generated ideal of D_P is a cancellation ideal and, so, D_P is a valuation domain [19, Theorem 24.3]. Therefore the semistar operation $\star$ is defined by a family of valuation overrings of D which are essential for D. We have already shown in Example 2 that, in this case, D is a $\star$–domain.

Conversely, if D is a $\star$–domain then $\star$ is a.b. (Proposition 3) and, as we already remarked, if $\star$ is a spectral semistar operation then $\star$ is stable.

The next proposition generalizes to the case of $\star$–domains some results already known for Prüfer $\star$–multiplication domains (cf. [13, Proposition 3.1 and 3.2]).

Proposition 6. *Let T be an overring of an integral domain D and let $\iota : D \hookrightarrow T$ be the canonical embedding.*

- **(1)** *Let $\star$ be a semistar operation on D and let $\star_\iota$ be the semistar operation on T defined by $E^{\star_\iota} := E^\star$, for each $E \in \overline{\boldsymbol{F}}(T)$ $(\subseteq \overline{\boldsymbol{F}}(D))$. If D is a $\star$–domain then T is a $\star_\iota$–domain.*
- **(2)** *Let $*$ be a semistar operation on T and let $*^\iota$ be the semistar operation on D defined by $E^{*^\iota} := (ET)^*$, for each $E \in \overline{\boldsymbol{F}}(D)$. If T is a $*$–domain and ι is flat then D is a $*^\iota$–domain.*

Proof. **(1)** Let $G := x_1T + x_2T + \cdots + x_nT \in \boldsymbol{f}(T)$ and set $G_0 := x_1D + x_2D + \cdots + x_nD$ $(\in \boldsymbol{f}(D))$. Then $(G(T : G))^{\star_\iota} = (G_0T(T : G_0T))^\star \supseteq ((G_0(D : G_0))T)^\star = ((G_0(D : G_0))^\star T)^\star = (D^\star T)^\star = T^\star$, thus we conclude immediately that $(G(T : G))^{\star_\iota}$ coincides with $T^{\star_\iota}$.

(2) Let $F := x_1D + x_2D + \cdots + x_nD \in \boldsymbol{f}(D)$. Then $(F(D : F))^{*^\iota} = ((F(D : F))T)^* = (FT(T : FT))^* = T^* = D^{*^\iota}$. □

Remark 4. Note that the semistar operation $v(D^\star)$ considered in Proposition 2 (5) coincides with $(v_{D^\star})^\iota$ (notation as in Proposition 6 (2)), where $\iota : D \hookrightarrow D^\star$ is the canonical embedding and $v_{D^\star}$ is the v–(semi)star operation on $D^\star$.

Example 3. *The assumption of flatness is essential in the proof of Proposition 6 (2).*

Let $k \subset K$ be a proper finite extension of fields and X an indeterminate over K. Set $T := K[X]_{(X)}$, $D := k + XK[X]_{(X)}$, $M := XK[X]_{(X)}$, $\iota : D \hookrightarrow T$ the canonical embedding (which is clearly non-flat). Note that T, being a discrete valuation domain, is a P$*$MD (and so a $*$–domain) for all the semistar operations $*$ on T, in particular T is a Pd_TMD, where d_T is the identity (semi)star operation on T. On the other hand D is not a $(d_T)^\iota$–domain, since $(D : M) = (M : M) = T$, hence $(MM^{-1})^{(d_T)^\iota} = (MT)^{(d_T)^\iota} = M \neq D$ and M is finitely generated in D, by the finiteness of ι [9, Proposition 1.8].

As we have already observed (Example 1 (1)), $(d_T)^\iota$ is an a.b. semistar operation on D, since $(d_T)^\iota = \star_{\{T\}}$ and T is a valuation domain; therefore $\star := (d_T)^\iota$ $(= \star_{\{T\}})$ gives an example of an a.b. semistar operation on D such that D is not a $\star$–domain. Moreover, $d_D = \widetilde{\star_{\{T\}}} = (\overline{\star_{\{T\}}})_f \lneq (\star_{\{T\}})_f = \star_{\{T\}}$, since $\star_{\{T\}}$ is stable if and only if $\iota : D \hookrightarrow T$ is flat (cf. [35, Proposition 1.7] and [32, Theorem 7.4 (i)]).

The next result shows that a $\star$–domain may be a $((\overline{\star})_f, \star_f)$–domain even if it is not integrally closed (cf. Proposition 5).

Corollary 3. *Let D be a $\star$-domain. Assume that $D^\star$ is flat over D. Then D is a $((\overline{\star})_f, \star_f)$–domain.*

Proof. Let $\iota : D \hookrightarrow D^\star$ be the canonical embedding. Then, $D^\star$ is a $\star_\iota$–domain, by Proposition 6 (1). Since $D^\star$ is integrally closed (Proposition 3), we can apply Proposition 5 and get that $D^\star$ is a $((\overline{\star_\iota})_f, (\star_\iota)_f)$–domain. By using also the flatness assumption of $D^\star$ over D, we have $(E \cap F)^\star = ((E \cap F)D^\star)^\star = (ED^\star \cap FD^\star)^\star = ((ED^\star)^\star \cap (FD^\star)^\star)^\star = ((ED^\star)^{\star_\iota} \cap (FD^\star)^{\star_\iota})^{\star_\iota} = (ED^\star)^{\star_\iota} \cap (FD^\star)^{\star_\iota} = (ED^\star)^\star \cap (FD^\star)^\star = E^\star \cap F^\star$, for all $E, F \in \boldsymbol{f}(D)$. The conclusion follows from Remark 3 (2). $\square$

We conclude this section with a transfer-type result.

Proposition 7. *Let D be an integral domain and $\star$ a semistar operation on D. Let $\iota : D \hookrightarrow D^\star$ be the canonical embedding. If D is a $((\overline{\star})_f, \star_f)$–domain then $D^\star$ is a $((\overline{\star_\iota})_f, (\star_\iota)_f)$–domain.*

Proof. We prove the claim by using the equivalence (i)⇔(ii) of Remark 3 (2). Let $E, F \in \boldsymbol{f}(D^\star)$. There exist $E_0, F_0 \in \boldsymbol{f}(D)$ such that $E = E_0D^\star$ and $F = F_0D^\star$. Then $(E \cap F)^{\star_\iota} = (E_0D^\star \cap F_0D^\star)^{\star_\iota} = (E_0D^\star \cap F_0D^\star)^\star \subseteq (E_0D^\star)^\star \cap (F_0D^\star)^\star = E_0^\star \cap F_0^\star = (E_0 \cap F_0)^\star \subseteq (E \cap F)^\star = (E \cap F)^{\star_\iota}$. Thus $(E \cap F)^{\star_\iota} = (E_0D^\star)^\star \cap (F_0D^\star)^\star = E^\star \cap F^\star = E^{\star_\iota} \cap F^{\star_\iota}$. $\square$

3. Prüfer $\star$–multiplication domains and $\star$-coherence

In this section, we look for conditions for a $\star$-domain to be a P$\star$MD, by using coherent-like conditions.

We say that a *domain* D is

a) $\star$*–extracoherent* if for all $E, F \in \boldsymbol{f}(D)$, with $0 \neq E \cap F$, there exists $J \in \boldsymbol{f}(D)$, with $J \subseteq E \cap F$, such that $J^\star = E^\star \cap F^\star$;

b) $\star$*–coherent* if for all $E, F \in \boldsymbol{f}(D)$, with $0 \neq E \cap F$, there exists $J \in \boldsymbol{f}(D)$, such that $J^\star = E^\star \cap F^\star$ (i.e. $E^\star \cap F^\star$ is $\star$–finite [17, page 650]);

c) *truly* $\star$*–coherent* if for all $E, F \in \boldsymbol{f}(D)$, with $0 \neq E \cap F$, there exists $J \in \boldsymbol{f}(D)$, such that $J^\star = (E \cap F)^\star$ (i.e. $E \cap F$ is $\star$–finite);

d) $\star$*–quasi-coherent* if for each $F \in \boldsymbol{f}(D)$, $(D : F)^\star = G^\star$ for some $G \in \boldsymbol{f}(D)$ (i.e. $(D : F)$ is $\star$–finite).

Remark 5. Note that, without loss of generality, the properties a), b), c) and d) can be tested for all $E', F' \in \boldsymbol{f}(D)$ and E', F' ideals in D. As a matter of fact, if $E, F \in \boldsymbol{f}(D)$, then for some nonzero elements $e, f \in D$, $eE, fF \subseteq D$, thus for $h := ef$ we have $E' := hE, F' := hF \in \boldsymbol{f}(D)$ and E', $F' \subseteq D$. Therefore if $J' \in \boldsymbol{f}(D)$ is such that $J'^\star = E'^\star \cap F'^\star$ [respectively: $J'^\star = (E' \cap F')^\star$], then $(h^{-1}J')^\star = E^\star \cap F^\star$ [respectively: $(h^{-1}J')^\star = (E \cap F)^\star$]. Moreover, $J' \subseteq E' \cap F'$, then $J := h^{-1}J' \subseteq h^{-1}E' \cap h^{-1}F' = h^{-1}hE \cap h^{-1}hF = E \cap F$. For d), for each $F \in \boldsymbol{f}(D)$, let $f \in D$ be a nonzero element of D such that $F' := fF \subseteq D$. If $J' \in \boldsymbol{f}(D)$ is such that $J'^\star = (D : F')^\star$, then it is easy to see that $J := f^{-1}J' \in \boldsymbol{f}(D)$ is such that $J^\star = (D : F)^\star$.

Recall that given a semistar operation $\star$ on an integral domain D, D is called a $\star$*–Noetherian domain* if D has the ascending chain condition on the quasi–$\star$–ideals (i.e. the nonzero ideals J of D such that $J = J^\star \cap D$), [8, Section 3].

Examples 1. (1) *An integral domain D and a (semi)star operation $\star$ such that D is $\star$–quasi-coherent but it is neither $\star$–coherent nor truly $\star$–coherent.*

For $\star = d$, the notions of $\star$–extracoherent domain, truly $\star$–coherent domain and $\star$–coherent domain coincide with the classical notion of coherent domain; the notion

of d–quasi-coherent domain coincides with the classical notion of quasi-coherent domain [5]. Therefore it is sufficient to take a quasi-coherent non-coherent domain (see [21, Examples 4.4 and 5.3]).

(2) *For $\star = v$, the notions of $\star$–coherent domain and $\star$–quasi-coherent domain coincide with the notion of v–coherent domain* [10, Proposition 3.6].

(3) *A $\star$–Noetherian domain (e.g. a Noetherian domain) is truly $\star$–coherent (and truly $\star_{_f}$–coherent)*

Recall that in a $\star$–Noetherian domain each nonzero fractional ideal is $\star_{_f}$–finite [8, Lemma 3.3], thus it is obvious that a $\star$–Noetherian domain is truly $\star$–coherent (or truly $\star_{_f}$–coherent).

(4) *A Noetherian domain (thus, in particular, a truly $\star$–coherent) is not necessarily a $\star$–extracoherent domain.* (This fact led us to use the terminology of "extracoherent" for this type of "strong $\star$–coherence", cf. also the following Theorem 1 (1).)

In order to construct an example of the type announced above, we start by recalling that, for $\star = v$, even when D is Noetherian, $(E \cap F)^v$ maybe properly included in $E^v \cap F^v$. An explicit example was constructed in [1, page 4] as follows. Let K be a field and X an indeterminate, set $D := K[\![X^3, X^4, X^5]\!]$, $E := (X^3, X^4)$, $F := (X^3, X^5)$, $M := (X^3, X^4, X^5)$. Note that $E^v = (D : (D : (X^3, X^4))) = (D : (X^{-3}D \cap X^{-4}D)) = (K[\![X^3, X^4, X^5]\!] : K[\![X]\!]) = (X^3, X^4, X^5) = M$; similarly $F^v = M$. Therefore $(X^3) = (X^3)^v = (E \cap F)^v \subsetneq E^v \cap F^v = (X^3, X^4)^v \cap (X^3, X^5)^v = M \cap M = M$.

(5) Note that, even if a coherent domain is not necessarily $\star$–extracoherent by (4), e.g. for $\star = v$, however it is an easy consequence of the definitions that *a coherent domain (e.g. a Prüfer domain* [19, Proposition 25.4 (1)]*) is $\star$–extracoherent domain, for each stable semistar operation $\star$.*

Lemma 1. *Let D be an integral domain and $\star$ a semistar operation on D. Then:*

- **(1)** *The $\star$–extracoherent domains coincide with the $\star_{_f}$–extracoherent domains and the $\star$–coherent domains coincide with the $\star_{_f}$–coherent domains.*
- **(2)** *D is a truly $\star_{_f}$–coherent domain if and only if, for all $E, F \in \boldsymbol{f}(D)$, there exists $J \in \boldsymbol{f}(D)$, with $J \subseteq E \cap F$, such that $J^\star = (E \cap F)^\star$ (or, equivalently, if $E \cap F$ is $\star_{_f}$–finite). In particular, a truly $\star_{_f}$–coherent domain is a truly $\star$–coherent domain.*
- **(3)** *D is a $\star_{_f}$–quasi-coherent domain if, for each $F \in \boldsymbol{f}(D)$, $(D : F)^\star = G^\star$ for some $G \in \boldsymbol{f}(D)$, with $G \subseteq (D : F)$ (or, equivalently, if $(D : F)$ is $\star_{_f}$–finite). In particular, a $\star_{_f}$–quasi-coherent domain is a $\star$–quasi-coherent domain.*

Proof. (1) follows immediately from the definitions. (2) and (3) are straightforward consequences of [17, Lemma 2.3]. □

Remark 6. If $\star$ is a (semi)star operation on a domain D, then, in particular, for each $F \in \boldsymbol{f}(D)$, $(D : F)$ is a divisorial ideal thus $(D : F) = (D : F)^\star = (D : F)^v$ [19, Theorem 34.1 (3, 4)], hence D is $\star_{_f}$–quasi-coherent if and only if D is $\star$–quasi-coherent (Lemma 1 (3)).

Proposition 8. *Let $\star$ be a semistar operation on an integral domain D. Assume that D is a $((\overline{\star})_{_f}, \star_{_f})$–domain (e.g. $\star$ is a stable semistar operation on D). Then:*

- **(1)** *If D is a $\star$–Noetherian domain then D is $\star$–extracoherent.*
- **(2)** *The notions of truly $\star$–coherent domain and $\star$–coherent domain coincide.*
- **(3)** *D is $\star$–quasi-coherent if and only if $(D^\star : F)$ is $\star$–finite for each $F \in \boldsymbol{f}(D)$.*
- **(4)** *$\star$–coherent implies $\star$–quasi-coherent.*

Proof. (1) Let $E, F \in \boldsymbol{f}(D)$, with $0 \neq E \cap F$. We have already observed that, in a $((\overline{\star})_{_f}, \star_{_f})$–domain, $E^\star \cap F^\star = (E \cap F)^\star$. Moreover, by the $\star$–Noetherianity, there exists $J \in \boldsymbol{f}(D)$ such that $J^\star = (E \cap F)^\star$ and $J \subseteq E \cap F$ [8, Lemma 3.3].

(2) and (3) are obvious since in this situation, for all $E, F \in \boldsymbol{f}(D)$, $(E \cap F)^\star = E^\star \cap F^\star$ (Remark 3 (2)); similarly, if $F = x_1D + x_2D + \cdots + x_nD$, in the present situation we have $(D : F)^\star = (x_1^{-1}D \cap x_2^{-1}D \cap \cdots \cap x_n^{-1}D)^\star = (x_1^{-1}D)^\star \cap (x_2^{-1}D)^\star \cap \cdots \cap (x_n^{-1}D)^\star = x_1^{-1}D^\star \cap x_2^{-1}D^\star \cap \cdots \cap x_n^{-1}D^\star = (D^\star : F^\star) = (D^\star : F)$.

(4) If $F = x_1D + x_2D + \cdots + x_nD$, then $(D : F)^\star = (x_1^{-1}D \cap x_2^{-1}D \cap \cdots \cap x_n^{-1}D)^\star = (x_1^{-1}D)^\star \cap (x_2^{-1}D)^\star \cap \cdots \cap (x_n^{-1}D)^\star$ and $(x_1^{-1}D)^\star \cap (x_2^{-1}D)^\star \cap \cdots \cap (x_n^{-1}D)^\star = G^\star$, for some $G \in \boldsymbol{f}(D)$, by the $\star$–coherence of D. $\square$

Theorem 1. *Let D be an integral domain and $\star, \star_1, \star_2$ semistar operations on D. Then:*

- **(1)** *If D is $\star$–extracoherent then D is $\star$–coherent and truly $\star$–coherent.*
- **(2)** *If D is truly $\star$–coherent then D is $\star$–quasi-coherent.*
- **(3)** *Assume that $\star_1 \leq \star_2$. If D is truly $\star_1$–coherent [respectively: $\star_1$–quasi-coherent] then D is truly $\star_2$–coherent, [respectively: $\star_2$–quasi-coherent]. Assume, moreover, that $\star_2$ is stable. If D is $\star_1$–extracoherent [respectively: $\star_1$–coherent] then D is $\star_2$–extracoherent [respectively: $\star_2$–coherent].*

Let ι be the canonical embedding of D in $D^\star$. Then:

- **(4)** *D is $\star$–coherent if and only if $D^\star$ is $\star_\iota$–coherent.*
- **(5)** *If D is $\star$–extracoherent then $D^\star$ is $\star_\iota$–extracoherent.*
- **(6)** *Assume, moreover that D is a $((\overline{\star})_{_f}, \star_{_f})$–domain (e.g. $\star$ is stable); then D is truly $\star$–coherent [respectively: $\star$–quasi-coherent] if and only if $D^\star$ is truly $\star_\iota$–coherent [respectively: $\star_\iota$–quasi-coherent].*

Proof. (1) follows from the definitions and from the fact that, in general, $(E \cap F)^\star \subseteq E^\star \cap F^\star$.

(2) Recall that, if $F = x_1D + x_2D + \cdots + x_nD$, then $(D : F)^\star = (x_1^{-1}D \cap x_2^{-1}D \cap \cdots \cap x_n^{-1}D)^\star$, thus truly $\star$–coherent implies $\star$–quasi-coherent.

(3) In general, it is easy to see that if $\star_1 \leq \star_2$ and if an ideal is $\star_1$–finite, it is also $\star_2$–finite. The second part of the statement follows from the fact that if $J \in \boldsymbol{f}(D)$ is such that $J^{\star_1} = E^{\star_1} \cap F^{\star_1}$, then $J^{\star_2} = (J^{\star_1})^{\star_2} = (E^{\star_1} \cap F^{\star_1})^{\star_2}$. By the stability of $\star_2$, we have $(E^{\star_1} \cap F^{\star_1})^{\star_2} = (E^{\star_1})^{\star_2} \cap (F^{\star_1})^{\star_2} = E^{\star_2} \cap F^{\star_2}$.

(4) Assume that D is $\star$–coherent. Let $E, F \in \boldsymbol{f}(D^\star)$ and let $E_0, F_0 \in \boldsymbol{f}(D)$ be such that $E = E_0D^\star$ and $F = F_0D^\star$. Then, there exists $J_0 \in \boldsymbol{f}(D)$ such that $J_0^\star = E_0^\star \cap F_0^\star$. Set $J := J_0D^\star \in \boldsymbol{f}(D^\star)$, then $J^{\star_\iota} = (J_0D^\star)^\star = J_0^\star = E_0^\star \cap F_0^\star = (E_0D^\star)^\star \cap (F_0D^\star)^\star = E^{\star_\iota} \cap F^{\star_\iota}$. Conversely, let $E, F \in \boldsymbol{f}(D)$. Then $ED^\star, FD^\star \in \boldsymbol{f}(D^\star)$. It follows that there exists $H \in \boldsymbol{f}(D^\star)$ such that $H^\star = H^{\star_\iota} = (ED^\star)^{\star_\iota} \cap (FD^\star)^{\star_\iota} = (ED^\star)^\star \cap (FD^\star)^\star$. Let $H_0 \in \boldsymbol{f}(D)$ such that $H = H_0D^\star$. Then $(H_0)^\star = (H_0D^\star)^\star = H^\star = (ED^\star)^\star \cap (FD^\star)^\star = E^\star \cap F^\star$.

(5) Let E, F, E_0, F_0, J_0 like in the first part of the proof of (4). Observe that, in this case, we can take $J_0 \subseteq E_0 \cap F_0$. Then $J = J_0D^\star \subseteq (E_0 \cap F_0)D^\star \subseteq E_0D^\star \cap F_0D^\star = E \cap F$ and we conclude like in the first part of the proof of (4).

(6) The statement for the truly coherent case follows from (4) and Proposition 8 (2), since $\star_\iota$ is a (semi)star operation on $D^\star$ and if D is a $((\overline{\star})_{_f}, \star_{_f})$–domain, then $D^\star$ is a $((\overline{\star_\iota})_f, (\star_\iota)_f)$–domain (Proposition 7).

Now suppose that D is $\star$–quasi-coherent. Let $F \in \boldsymbol{f}(D^\star)$ and let $F_0 \in \boldsymbol{f}(D)$ be such that $F = F_0D^\star$. We know that there exists $G_0 \in \boldsymbol{f}(D)$ such that $(D : F_0)^\star = G_0^\star$. Then by the assumption we have $(D^\star : F_0D^\star)^\star = (D : F_0)^\star$. The conclusion is now straightforward.

Conversely, assume that D is is $\star_\iota$–quasi-coherent. Let $F_0 = x_1D + x_2D + \cdots + x_nD \in \boldsymbol{f}(D)$. For $F := F_0D^\star \in \boldsymbol{f}(D^\star)$, we know that there exists $G_0 \in \boldsymbol{f}(D)$

such that $G := G_0 D^\star \in \boldsymbol{f}(D^\star)$ has the property that $(D^\star : F_0 D^\star)^\star = (D^\star : F)^{\star_\iota} = G^{\star_\iota} = (G_0 D^\star)^{\star_\iota} = G_0^\star$. Since D is a $((\overline{\star})_{_f}, \star_{_f})$–domain, we have that $(D^\star : F_0 D^\star) = (D^\star : F_0) = (D^{\star_f} : (x_1 D + x_2 D + \cdots + x_n D)) = x_1^{-1} D^{\star_f} \cap x_2^{-1} D^{\star_f} \cap \cdots \cap x_n^{-1} D^{\star_f} = (x_1^{-1} D \cap x_2^{-1} D \cap \cdots \cap x_n^{-1} D)^{\star_f} = (D : F_0)^{\star_f}$, therefore $(D^\star : F_0 D^\star)^\star = ((D : F_0)^{\star_f})^\star = (D : F_0)^\star$, thus we can conclude that D is $\star$–quasi-coherent. □

Examples 2. (1) *A $\star$–Noetherian domain (thus, in particular by Example 1 (3), a truly $\star$–coherent domain) is not necessarily $\star$–coherent.*

Let D be a 2-dimensional Noetherian domain. The integral closure D' of D is a 2-dimensional Krull domain. Clearly D' is not a Prüfer domain (since a Prüfer Krull domain is a Dedekind domain [19, Theorem 43.16] and so 1-dimensional). Thus, by [20, Theorem 5.3.15], there exists a proper overring T of D that is not coherent. Consider the semistar operation $\star_{\{T\}}$ on D. Since D is Noetherian, it is obviously $\star_{\{T\}}$–Noetherian. Let ι be the canonical embedding of D in T. We have that D is not $\star_{\{T\}}$–coherent, otherwise T would be $(\star_{\{T\}})_\iota$–coherent (Theorem 1 (4)). This is impossible, since $(\star_{\{T\}})_\iota = d_T$, and d_T–coherent means coherent.

(2) *A v-coherent domain is not necessarily v–extracoherent.*

Note that, since a $\star$–Noetherian domain is truly $\star$–coherent (Example 1 (3)), it is also $\star$–quasi-coherent (Theorem 1 (2)). Therefore, taking $\star = v$, a v–Noetherian domain (that is, a Mori domain) is v–quasi-coherent or, equivalently, v–coherent (Example 1 (2)). It follows that the Noetherian (in particular, v–Noetherian) domain constructed in Example 1 (4) is a v–coherent domain which is not v–extracoherent (and so, the notion of $\star$–coherence and $\star$–extracoherence are distinct).

Corollary 4. *Let D be an integral domain and $\star$ a semistar operation on D. The following statements are equivalent:*

- **(i)** *D is $\widetilde{\star}$–extracoherent.*
- **(ii)** *D is truly $\widetilde{\star}$–coherent.*
- **(iii)** *D is $\widetilde{\star}$–coherent.*

Proof. We know already that (i)⇒(ii)⇔(iii) (Proposition 8 (2) and Theorem 1 (1)).

Assume that D is truly $\widetilde{\star}$–coherent. Since $\widetilde{\star}$ is a semistar operation stable and of finite type, then for all $E, F \in \boldsymbol{f}(D)$, there exists $J \in \boldsymbol{f}(D)$ such that $J \subseteq E \cap F$ and $J^{\widetilde{\star}} = (E \cap F)^{\widetilde{\star}} = E^{\widetilde{\star}} \cap F^{\widetilde{\star}}$ [17, Lemma 2.3]. □

The next goal is to characterize the $\star$–extracoherence by using the other weaker $\star$–coherence-like conditions. We start with an useful lemma

Lemma 2. *Let D be an integral domain and $\star$ a semistar operation on D. Assume that D is $\star$–extracoherent. Then D is a $((\overline{\star})_{_f}, \star_{_f})$–domain (or, equivalently, D is a $(\widetilde{\star}, \star_{_f})$–domain).*

Proof. Let $E, F \in \boldsymbol{f}(D)$. Then there exists $J \in \boldsymbol{f}(D)$, $J \subseteq E \cap F$, such that $J^\star = E^\star \cap F^\star$. Moreover, obviously, $J^\star \subseteq (E \cap F)^\star \subseteq E^\star \cap F^\star$. Hence, in particular, $(E \cap F)^\star = E^\star \cap F^\star$, thus we conclude by Remark 3 (2, (i)⇔(ii)).

For the parenthetical statement, note that $\widetilde{\star} \leq (\overline{\star})_{_f} \leq \star_{_f}$, thus in the present situation we need only to prove that $((\overline{\star})_{_f}, \star_{_f})$–domain implies $(\widetilde{\star}, \star_{_f})$–domain). Since a $\star$–extracoherent domain is also $\star_{_f}$–extracoherent (Lemma 1 (1)), we have that in a $\star_{_f}$–extracoherent domain $\widetilde{\star} = (\overline{\star_{_f}})_{_f} = \star_{_f}$, by what we have already proved and Remark 3 (3). □

Remark 7. By using Lemma 2, we can easily improve the result in Example 1 (5) and obtain that: *A coherent domain is $\star$–extracoherent if and only if it is a $((\overline{\star})_{_f}, \star_{_f})$–domain.* A more precise statement will be proved in the following Proposition 9.

Proposition 9. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

- **(i)** *D is $\star$–extracoherent.*
- **(i$_f$)** *D is $\star_f$–extracoherent.*
- **$(\widetilde{\text{i}})$** *D is $\widetilde{\star}$–extracoherent and a $(\widetilde{\star}, \star_f)$–domain.*
- **(ii)** *D is truly $\star$–coherent and a $(\widetilde{\star}, \star_f)$–domain.*
- **(ii$_f$)** *D is truly $\star_f$–coherent and a $(\widetilde{\star}, \star_f)$–domain.*
- **$(\widetilde{\text{ii}})$** *D is truly $\widetilde{\star}$–coherent and a $(\widetilde{\star}, \star_f)$–domain.*
- **(iii)** *D is $\star$–coherent and a $(\widetilde{\star}, \star_f)$–domain.*
- **(iii$_f$)** *D is $\star_f$–coherent and $(\widetilde{\star}, \star_f)$–domain.*
- **$(\widetilde{\text{iii}})$** *D is $\tilde{\star}$–coherent and $(\widetilde{\star}, \star_f)$–domain.*

Proof. The equivalences (i)$\Leftrightarrow$(i$_f$) and (iii)$\Leftrightarrow$(iii$_f$) are in Lemma 1 (1).

(ii$_f$)$\Leftrightarrow$$(\widetilde{\text{ii}})$, (iii$_f$)$\Leftrightarrow$$(\widetilde{\text{iii}})$ and $(\widetilde{\text{i}})$$\Rightarrow$(i$_f$) are trivial. Note also that $(\widetilde{\text{i}})$$\Leftrightarrow$$(\widetilde{\text{ii}})$$\Leftrightarrow$$(\widetilde{\text{iii}})$ by Corollary 4.

(ii)$\Leftrightarrow$(ii$_f$). Observe that, when $\star_f = \widetilde{\star}$, for all $E, F \in \boldsymbol{f}(D)$, we have $(E \cap F)^{\star_f} \subseteq (E \cap F)^{\star} \subseteq E^{\star} \cap F^{\star} = E^{\star_f} \cap F^{\star_f} = (E \cap F)^{\star_f}$. Since in the present situation $(E \cap F)^{\star_f} = (E \cap F)^{\star}$, it is clear that the notions of truly $\star$–coherent and truly $\star_f$–coherent are equivalent.

(i$_f$)$\Leftrightarrow$(ii$_f$). By the previous considerations, we already know that (ii$_f$)($\Leftrightarrow$ $(\widetilde{\text{ii}})$ $\Leftrightarrow$ $(\widetilde{\text{i}})$) $\Rightarrow$(i$_f$).

Conversely, if D is $\star_f$–extracoherent, then D is truly $\star_f$-coherent by Theorem 1 (1) and a $(\widetilde{\star}, \star_f)$–domain by Lemma 2 (and by (i)$\Leftrightarrow$(i$_f$)). □

The next goal is characterize the Prüfer $\star$–multiplication domains among the $\star$–domains using coherence-like conditions.

Proposition 10. *Let D be an integral domain and $\star$ a semistar operation on D. If D is a P$\star$MD then D is $\tilde{\star}$–extracoherent.*

Proof. We claim that, in a P$\star$MD, for all $E, F \in \boldsymbol{F}(D)$

$$((E+F)(E \cap F))^{\tilde{\star}} = (EF)^{\tilde{\star}}.$$

Indeed, let $M \in \mathcal{M}(\star_f)$. Then D_M is a valuation domain and so $(E+F)(E \cap F)D_M = (ED_M + FD_M)(ED_M \cap FD_M) = EFD_M$, by [19, Theorem 25.2 (d) and Remark 25.3]. By the definition of $\widetilde{\star}$, we deduce the claim.

Now, if $E, F \in \boldsymbol{f}(D)$, EF is finitely generated, thus $(EF(D : EF))D_M = D_M$, for each $M \in \mathcal{M}(\star_f)$ and so we obtain that EF is $\widetilde{\star}$–invertible [17, Theorem 2.23].

Therefore, $(E \cap F)$ is also $\widetilde{\star}$–invertible [17, Lemma 2.1(2)] and, hence, $\widetilde{\star}$–finite, i.e. $J^{\widetilde{\star}} = (E \cap F)^{\widetilde{\star}} = E^{\widetilde{\star}} \cap F^{\widetilde{\star}}$, for some $J \in \boldsymbol{f}(D)$, with $J \subseteq E \cap F$ [17, Lemma 2.3 and Proposition 2.6]. □

Remark 8. Since in a P$\star$MD it is known that $\widetilde{\star} = \star_f$ [13, Theorem 3.1 ((v)$\Leftrightarrow$(vi))], then from the previous Proposition 10 and Theorem 1 (1), we deduce that a P$\star$MD is truly $\widetilde{\star}$–coherent. Therefore, from Proposition 9, we have that a P$\star$MD is also $\star$–extracoherent, $\star_f$–extracoherent, truly $\star$–coherent, truly $\star_f$–coherent, $\star$–coherent, $\star_f$–coherent, and $\widetilde{\star}$–coherent. Moreover, it is also $\star$–quasi-coherent, $\star_f$–quasi-coherent and $\widetilde{\star}$–quasi-coherent, by Theorem 1 (2).

Note also that, from the fact that $\widetilde{\star} \leq \overline{\star} \leq \star$, $\widetilde{\star} \leq \star_f$ and that $\overline{\star}$ is stable, it follows that a P$\star$MD is $\overline{\star}$–extracoherent (Proposition 10 and Theorem 1 (3)) and so truly $\overline{\star}$–coherent, $\overline{\star}$–coherent and $\overline{\star}$–quasi-coherent (Theorem 1 (1, 2)).

We are now in condition of proving the main theorem of this section.

Theorem 2. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

- **(i)** *D is a P$\star$MD.*
- **(ii)** *D is a $\star$–extracoherent $\star$–domain.*
- **($\mathbf{ii}_{\boldsymbol{f}}$)** *$D$ is a $\star_{_f}$–extracoherent $\star$–domain.*
- **($\widetilde{\mathbf{ii}}$)** *D is a $\widetilde{\star}$–extracoherent $\star$–domain.*
- **($\mathbf{iii}_{\boldsymbol{f}}$)** *$D$ is a truly $\star_{_f}$–coherent $\star$–domain.*
- **($\widetilde{\mathbf{iii}}$)** *D is a truly $\widetilde{\star}$–coherent $\star$–domain.*
- **($\widetilde{\mathbf{iv}}$)** *D is a $\widetilde{\star}$–coherent $\star$–domain.*
- **($\mathbf{v}_{\boldsymbol{f}}$)** *$D$ is a $\star_{_f}$–quasi-coherent $\star$–domain (or, equivalently, a $\star_{_f}$–domain).*
- **($\widetilde{\mathbf{v}}$)** *D is a $\widetilde{\star}$–quasi-coherent $\star$–domain (or, equivalently, a $\widetilde{\star}$–domain).*

In particular, a quasi-coherent $\star$–domain is a P$\star$MD.

Proof. (i)$\Rightarrow$($\widetilde{\text{ii}}$)$\Leftrightarrow$($\widetilde{\text{iii}}$)$\Leftrightarrow$ ($\widetilde{\text{iv}}$)$\Rightarrow$($\widetilde{\text{v}}$)$\Rightarrow$(v$_f$) by Proposition 2 (3), Proposition 8 (2), Corollary 4, Proposition 10 and Theorem 1 (1, 2, 3).

(i)$\Rightarrow$(ii)$\Leftrightarrow$(ii$_f$) by Proposition 2 (3), Remark 8 and Lemma 1 (1).

(ii)$\Rightarrow$(iii), (ii)$\Rightarrow$(iv)$\Leftrightarrow$(iv$_f$) and (ii$_f$)$\Rightarrow$(iii$_f$)$\Rightarrow$(v$_f$) by Theorem 1 (1, 2) and Lemma 1 (1).

(v$_f$)$\Rightarrow$(i) Let $F \in \boldsymbol{f}(D)$. Then $(FF^{-1})^\star = D^\star$, since D is a $\star$–domain. By the fact that D is $\star_{_f}$–quasi-coherent, we can find $G \in \boldsymbol{f}(D)$, with $G \subseteq (D : F)$, such that $G^{\star_f} = (F^{-1})^{\star_f}$ (Lemma 1 (3)). Since $FG \in \boldsymbol{f}(D)$ and $G \subseteq F^{-1}$, then we have $D^\star = (FF^{-1})^\star = (F(F^{-1})^{\star_f})^\star = (FG^{\star_f})^\star = (FG)^\star = (FG)^{\star_f} \subseteq (FF^{-1})^{\star_f} \subseteq D^\star$. Therefore F is $\star_{_f}$-invertible, and so D is a P$\star$MD.

For the parenthetical statements in (v$_f$) and ($\widetilde{\text{v}}$) see Proposition 2 (2) and Lemma 1 (3). The last claim follows from Theorem 1 (3) and (v$_f$)$\Rightarrow$(i). □

Remark 9. **(1)** Note that, by Remark 6, Theorem 1 (1) and the previous theorem, *when $\star$ is a (semi)star operation on D, then the following conditions are equivalent to each of the statements in Theorem 2:*

- **(iii)** *D is a truly $\star$–coherent $\star$–domain.*
- **(iv)** *D is a $\star$–coherent (or, equivalently $\star_{_f}$–coherent, by Lemma 1 (1)) $\star$–domain.*
- **(v)** *D is a $\star$–quasi-coherent $\star$–domain.*

At the moment, we are unable to establish if the previous statements are also equivalent in the general semistar setting. However, if D is integrally closed or if $D^\star$ is flat over D and if $(\overline{\star})_{_f}$ is stable, then (iii)$\Leftrightarrow$(iv) (and they are equivalent to each of the statements of Theorem 2) by Proposition 5, Corollary 3, Lemma 2 and Proposition 9.

(2) From (1), in particular, we reobtain the following result proved by Gabelli and Houston [18, Proposition 3.2]:

$$D \text{ is a P}v\text{MD} \;\Leftrightarrow\; D \text{ is a } v\text{–(quasi-)coherent } v\text{–domain}\,.$$

Note that a similar characterization was given by Mott and Zafrullah in [33, Theorem 3.3]: D is a PvMD if and only if D is essential and, for all nonzero $a, b \in D$ $aD \cap bD$ is finitely generated. (Recall that an essential domain is a v–domain by [19, Proposition 44.13] and Remark 1 (5). On the other hand, the condition called *finite conductor* (for short, (FC)), i.e. for all nonzero $a, b \in D$ $aD \cap bD$ is finitely generated is *technically* weaker than the condition of quasi-coherence. However, in an essential domain, the condition (FC) is equivalent to the v–(quasi-)coherence [37, Lemma 8].)

Recall that a P$\star$MD which is a $\star$–Noetherian domain is called a $\star$–*Dedekind domain* [8, Proposition 4.1].

Corollary 5. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

- **(i)** *D is $\star$–Dedekind.*
- **(ii)** *D is a $\star$–Noetherian $\star$–domain.*

Proof. (i)$\Rightarrow$(ii) is obvious since a P$\star$MD is a $\star$–domain (Proposition 2 (3)).

(ii)$\Rightarrow$(i) Since a $\star$–Noetherian domain is truly $\star_f$–coherent (Example 1 (3)), we can apply Theorem 2 ((iii$_f$)$\Rightarrow$(i)). □

Examples 3. (1) *A (semi)star operation $\star$ on an integral domain D such that D is $\widetilde{\star}$–extracoherent but not $\star$–Noetherian (and, so, not $\widetilde{\star}$–Noetherian* [8, Remark 3.6 (2)]*).*

Take a PvMD non-Krull (or, equivalently, non-Mori [29, Theorem 3.2]) domain D (an explicit example is given next in Example 5). Then D is a w–extracoherent v–domain (Theorem 2), but D is not v–Noetherian, since a v–Noetherian v–domain is a v–Dedekind domain (Corollary 5) and v–Dedekind coincides with Krull [8, Remark 4.2 (1)].

(2) *A (semi)star operation $\star$ on an integral domain D such that D is not $\star$–quasi-coherent.*

For $\star = v$, take a v–domain D which is not a PvMD (Remark 1 (2)) then D is not a t–quasi-coherent (Theorem 2 ((v$_f$)$\Rightarrow$(i))) or, equivalently, D is not v–quasi-coherent (Remark 6).

For $\star = d$, take any integrally closed non-PvMD D and apply [37, Theorem 2] to conclude that D is not quasi–coherent (in fact, D does not verify (FC)).

Remark 10. (1) In this section we have introduced a variety of coherence-like definitions for a semistar operation. But we were mainly interested in a “strong form” of semistar coherence, that we have called “semistar–extracoherence”, for obtaining a characterization of the P$\star$MDs in terms of coherence-like conditions (cf. Proposition 10, Theorem 2).

We have also seen that all the possibly different coherence-like notions introduced here coincide for a $\star$–domain, when $\star$ is a (semi)star operation (Theorem 2 and Remark 9 (1)). However, it seems to us that it would be interesting to investigate further this subject in the general semistar setting and to study the interconnections among the various coherence-like conditions in some relevant situation (see, for instance, the following point (3)).

(2) Note that we have introduced a notion of “$\star$–coherence”, in order to be consistent with the definition of v–coherence already in the literature [10], but we believe that the “right” definition of coherence in the semistar setting is what we called “truly $\star$–coherence”. One of the reasons is that a $\star$–Noetherian domain is truly $\star$–coherent but, in general, it is not $\star$–coherent (Examples 1 (3) and 2 (1)). On the other hand, the fact that the notion of v–coherence works well in many situations is due to the fact that it coincides with the v–quasi-coherence.

Recall that we have already shown that, in general, a truly $\star$–coherent domain is not a $\star$–coherent domain (Example 2 (1)). The following Example 4 shows conversely that a $\star$–coherent domain is not a truly $\star$–coherent domain.

(3) In this circle of ideas, an open problem is related to the specific cases of v–, t– and w–operations. Note that, by Remark 6, Proposition 1 (1) and [10, Proposition 3.6], we know already that:

$$v\text{–coherent} \Leftrightarrow v\text{–quasi-coherent} \Leftrightarrow t\text{–quasi-coherent} \Leftrightarrow t\text{–coherent}.$$

Therefore, by Proposition 8 (2) and Theorem 1 (2), we have:

$$w\text{–extracoherent} \Leftrightarrow \text{(truly) } w\text{–coherent} \Rightarrow w\text{–quasi-coherent} \Rightarrow t\text{–quasi-coherent}$$

and, by Theorem 1 (3),

(truly) w–coherent $\Rightarrow$ truly t–coherent $\Rightarrow$ truly v–coherent $\Rightarrow$ v–(quasi-)coherent.

(Q-2) *Is it possible to give examples for showing that the previous implications do not invert ?*

We end this section with the example announced in the previous remark.

Example 4. *A $\star$–coherent domain which is not truly $\star$–coherent (hence, not $\star$–extracoherent), for an a.b. semistar operation $\star$ of finite type.*

Let $k \subset K$ be a proper extension of fields and let V a valuation domain of the form $K+M$, where M is the nonzero maximal ideal of V such that $M = M^2$. (For instance, take V to be a 1-dimensional nondiscrete valuation domain, having value group $\mathbb{R}$. Since $\mathbb{R} = 2\mathbb{R}$, then clearly $M = M^2$. To produce examples of dimension greater than 1, take V having value group equal to the lexicographically ordered direct product $\mathbb{R}^n$, with $n \geq 2$.) Set $D := k + M$.

Let $\star := \star_{\{V\}}$. Clearly $\star$ is an a.b. semistar operation of finite type on D. We claim that D is not coherent but it is $\star$–coherent.

Take $x \in K \setminus k$ and $m \in M$, $m \neq 0$, then we have $mD \cap mxD = mM$. In order to prove this equality, we note that x is a unit in V and M is a common ideal in D and V, thus obviously $xM = M$, and moreover $mM = mM \cap mxM \subseteq mD \cap mxD$. On the other hand, if $y \in mD \cap mxD$, then $y = m(h_1 + m_1) = mx(h_2 + m_2)$, with $h_1, h_2 \in k$ and $m_1, m_2 \in M$. Therefore $h_1 - xh_2 = xm_2 - m_1 \in M$, and so $h_1 - xh_2 = 0$. Since $x \in K \setminus k$, then necessarily $h_2 = 0 = h_1$ and thus $y = mm_1 \in mM$.

The equality $mD \cap xmD = mM$ and the fact that M is not finitely generated (since $M = M^2 \neq 0$), implies that D is not coherent (in fact, D is not a finite conductor domain). However, $(mD)^\star \cap (xmD)^\star = mV \cap xmV = mV \cap mV = mV = (mD)^\star$. More generally if $E, F \in \boldsymbol{f}(D)$ then EV and FV are principal and either $EV \cap FV = EV$ or $EV \cap FV = FV$, thus D is clearly $\star$–coherent. But D is not truly $\star$–coherent, since $(mD \cap xmD)^\star = (mM)^\star = mM$, thus $FV = F^\star \neq (mD \cap xmD)^\star = mM$, for each $F \in \boldsymbol{f}(D)$.

4. Prüfer $\star$–multiplication domains and $\mathtt{H}(\star)$–domains

We recall from [17, p. 651] that a domain D with a semistar operation $\star$ is an *$\mathtt{H}(\star)$–domain* if for each nonzero integral ideal I of D such that $I^\star = D^\star$ there exists a nonzero finitely generated ideal J, with $J \subseteq I$, such that $J^\star = D^\star$ (i.e. I is $\star_f$–finite), in other words D is called an $\mathtt{H}(\star)$–domain if $\mathcal{F}^\star = \mathcal{F}^{\star_f}$.

When $\star = v$, the $\mathtt{H}(v)$–domains coincide with the $\mathtt{H}$–domains introduced by Glaz and Vasconcelos [22, Remark 2.2 (c)].

It is obvious that each domain is an $\mathtt{H}(\star_f)$–domain, so the notion of $\mathtt{H}(\star)$–domain takes interest only when $\star$ is not of finite type.

In the next Proposition 11 we collect some characterizations of the $\mathtt{H}(\star)$–domains. Clearly a $\star$–Noetherian domain is an $\mathtt{H}(\star)$–domain [8, Lemma 3.3], thus we obtain in particular that Mori domains (e.g. Noetherian and Krull domains) are $\mathtt{H}$–domains. Houston and Zafrullah [28, Proposition 2.4] proved, more generally, that each (t, v)–domain (or TV-domain in their terminology) is an $\mathtt{H}$–domain. Note that a general class of $\mathtt{H}$–domains which are not (t, v)–domains was given in [28, Remark 2.5].

Example 5. *An $\mathtt{H}(\star)$–domain and P$\star$MD which is not a $\star$–Noetherian domain.*

Clearly, for $\star = d$, a Prüfer non-Dedekind domain provides an example of the type announced above.

A more elaborate example can be obtained by taking $\star = v$. Let K be a field and X, Y indeterminates over K. Set $D := K[X] + YK(X)[Y]_{(Y)}$. By the properties of the pullback constructions [9], D is a 2-dimensional Prüfer domain with infinitely many maximal ideals, each of them invertible (and so divisorial) and with a unique height 1 prime ideal $P := YK(X)[Y]_{(Y)}$ which is also divisorial (since $P = (D : T)$, where $T := K(X)[Y]_{(Y)}$) and it is contained in all the maximal ideals of D. Clearly, in this case $\mathcal{F}^v = \mathcal{F}^t = \{D\}$, thus D is an $\mathtt{H}(v)$–domain. However, D is not a Mori domain (in a Mori domain each nonzero element is contained in finitely many maximal t–ideals [6, Proposition 2.2], [7, Théorème 4.2]) and a Mori domain is precisely a v–Noetherian domain [8, Section 3] and [4, Theorem 2.1]. Note also that D provides an explicit example of an $\mathtt{H}(v)$–domain which is not a (t, v)–domain [28, Theorem 1.3 and Remark 2.5].

Proposition 11. *Let $\star$ be a semistar operation on an integral domain D. The following conditions are equivalent:*

- **(i)** *D is an $\mathtt{H}(\star)$–domain.*
- **(ii)** *Each quasi–$\star_f$–maximal ideal of D is a quasi–$\star$–ideal of D.*
- **(iii)** *For each $I \in \boldsymbol{F}(D)$, I is $\star$–invertible if and only if I is $\star_f$–invertible.*
- **(iii')** *For each $I \in \boldsymbol{F}(D)$, if I is $\star$–invertible then I and I^{-1} are $\star_f$–finite.*
- **(iv)** $\mathcal{M}(\star_f) = \mathcal{M}(\star)$.
- **(v)** $\mathcal{M}(\widetilde{\star}) = \mathcal{M}(\star)$.
- **(vi)** *The localizing system $\mathcal{F}^\star$ is finitely generated (i.e., $(\overline{\star})_f = \overline{\star}$* [11, Proposition 3.2]*).*
- **(vii)** $\widetilde{\star} = \overline{\star}$ *(i.e. D is a $(\widetilde{\star}, \overline{\star})$–domain).*
- **(viii)** $\overline{\star} \leq \star_f$.
- **(ix)** *For each nonzero prime ideal P of D such that $P^\star = D^\star$ there exists a nonzero finitely generated ideal J, with $J \subseteq P$, such that $J^\star = D^\star$ (i.e. P is $\star_f$–finite).*
- **(x)** *D is an $\mathtt{H}(\overline{\star})$–domain.*

Proof. (i)$\Leftrightarrow$(ii) [17, Lemma 2.7]. (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) [17, Proposition 2.8].

(i)$\Leftrightarrow$(vi) By definition, the localizing system $\mathcal{F}^\star$ is the set of the ideals I of D such that $I^\star = D^\star$. So, the definition of $\mathtt{H}(\star)$–domain is equivalent to require that $\mathcal{F}^\star$ is finitely generated.

(vi)$\Rightarrow$(vii) Recall that $\widetilde{\star} = \star_{(\mathcal{F}^\star)_f}$ (Proposition 1 (7)). It is clear that, if $\mathcal{F}^\star$ is finitely generated, then $\mathcal{F}^\star = (\mathcal{F}^\star)_f$ [11, Lemma 3.1]. Therefore $\widetilde{\star} = \star_{(\mathcal{F}^\star)_f} = \star_{\mathcal{F}^\star} = \overline{\star}$.

(vii)$\Rightarrow$(viii) It is clear, since $\widetilde{\star} \leq \star_f$.

(viii)$\Rightarrow$(iii) We have $\overline{\star} \leq \star_f \leq \star$, thus $\mathcal{F}^\star \subseteq \mathcal{F}^{\star_f} \subseteq \mathcal{F}^{\overline{\star}}$. Since $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}}$, then obviously $\mathcal{F}^\star = \mathcal{F}^{\star_f}$. Therefore, for an ideal $I \in \boldsymbol{F}(D)$, we have $II^{-1} \in \mathcal{F}^\star$ if and only if $II^{-1} \in \mathcal{F}^{\star_f}$; the conclusion now is straightforward.

(vii)$\Leftrightarrow$(x) Since $\overline{\overline{\star}} = \overline{\star}$ and $\widetilde{\overline{\star}} = \widetilde{\star}$ [11, Corollary 2.11 and Corollary 3.9].

(i)$\Rightarrow$(ix) is obvious.

(ix)$\Rightarrow$(i) Assume that D is not an $\mathtt{H}(\star)$-domain, thus $\mathcal{F}^{\star_f} \subsetneq \mathcal{F}^\star$. It is easy to see that the nonempty set $\mathcal{S} := \mathcal{F}^\star \setminus \mathcal{F}^{\star_f}$ is inductive. Let Q be a maximal element in $\mathcal{S}$. We claim that Q is a prime ideal of D. Suppose that x, y are two elements in $D \setminus Q$ such that $xy \in Q$. Then, by the maximality of Q in $\mathcal{S}$, we can find two finitely generated ideals of D, $J' \subseteq Q + xD$ and $J'' \subseteq Q + yD$ such that $J'^{\star_f} = D^\star$ and $J''^{\star_f} = D^\star$. On the other hand $J'J'' \subseteq Q^2 + xQ + yQ + xyD \subseteq Q$, and $(J'J'')^{\star_f} = (J'^{\star_f} J''^{\star_f})^{\star_f} = (D^\star)^\star = D^\star$, that contradicts the fact that $Q \in \mathcal{S}$.

Since Q is a prime ideal and $Q \in \mathcal{S} \subset \mathcal{F}^\star$, then by assumption there exists a nonzero finitely generated ideal $J \subseteq Q$ such that $J^\star = Q^\star = D^\star$, i.e. $Q^{\star_f} = D^\star$ or equivalently $Q \in \mathcal{F}^{\star_f}$, which is again a contradiction. $\square$

Corollary 6. *Let D be an integral domain and $\star$ a semistar operation on D. If D is an $\mathtt{H}(\star)$–domain the following conditions are equivalent:*

(i) *D is a $\star$-domain.*
(ii) *D is a P$\star$MD.*

Proof. It is a straightforward consequence of Proposition 11 ((i)$\Rightarrow$(iii)). $\square$

Note that a P$\star$MD is not always an $\mathtt{H}(\star)$–domain as the following example shows.

Example 6. *A (semi)star operation $\star$ on an integral domain D such that D is a P$\star$MD and a $((\overline{\star})_f, \star_f)$–domain but not an $\mathtt{H}(\star)$–domain and for which $\widetilde{\star} = \overline{\star_f} = (\overline{\star})_f = \star_f \lneq \overline{\star}$.*

Take a valuation domain V with a non-divisorial maximal ideal M (e.g. a rank 1 non-discrete valuation domain) and take $\star = v$. Clearly V is a PvMD, but not an $\mathtt{H}(v)$–domain, since the maximal ideal M is a t–ideal, but not divisorial (Proposition 11 ((i) $\Rightarrow$(iv))). Note that in this case $d = w$ $(= \widetilde{v}) = t$ $(= v_f)$, thus $d = (\overline{v})_f = t = \overline{v_f}$. Moreover, in a valuation domain, it is obvious that every (semi)star operation is stable, thus in particular $\overline{v} = v$. Finally, by the previous considerations, it follows that $(d =)\ t \lneq \overline{v}\ (= v)$, since $M \subsetneq M^v = V$.

Example 6 shows that the condition of being an $\mathtt{H}(\star)$–domain is too strong to turn a $\star$–domain into a P$\star$MD. On the other hand, the condition of being an $\mathtt{H}(\star)$–domain is equivalent to the fact that the subset $\mathrm{Inv}(D, \star_f)$ of $\mathrm{Inv}(D, \star)$ coincides with $\mathrm{Inv}(D, \star)$ (Proposition 11 ((i)$\Leftrightarrow$(iii))). We can weaken condition (iii) of the previous Proposition 11 and we call *$\mathtt{I}(\star)$–domain* an integral domain D such that $\mathrm{Inv}(D, \star) \cap \boldsymbol{f}(D) = \mathrm{Inv}(D, \star_f) \cap \boldsymbol{f}(D)$. Obviously an $\mathtt{H}(\star)$–domain is an $\mathtt{I}(\star)$–domain and if $\star$ is a semistar operation of finite type on an integral domain D, then D is always an $\mathtt{I}(\star)$–domain.

Proposition 12. *Let D be an integral domain and $\star$ a semistar operation on D. The following conditions are equivalent:*

(i) *D is an $\mathtt{I}(\star)$–domain.*
(ii) *If $F \in \boldsymbol{f}(D)$ is $\star$–invertible then F^{-1} is $\star_f$–finite.*
(iii) *D is an $\mathtt{I}(\overline{\star})$–domain.*

In particular, if D is $\star_f$–quasi-coherent then D is an $\mathtt{I}(\star)$–domain.

Proof. (i)$\Rightarrow$(ii) If F is $\star$–invertible, then it is $\star_f$–invertible. So, F^{-1} is $\star_f$–finite by [17, Proposition 2.6].

(ii)$\Rightarrow$(i) It follows easily from [17, Proposition 2.6].

(iii)$\Leftrightarrow$(i). Recall that a fractionary ideal I is $\star$–invertible if and only if is $\overline{\star}$–invertible, since $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}}$. By the equivalence (i)$\Leftrightarrow$(ii), we need to prove that if $F \in \boldsymbol{f}(D)$ is $\star$–invertible then F^{-1} is $\star_f$–finite if and only if F^{-1} is $(\overline{\star})_f$–finite.

Let F^{-1} be $(\overline{\star})_f$–finite. If $G \in \boldsymbol{f}(D)$ is such that $G \subseteq (D : F)$ and $G^{\overline{\star}} = (D : F)^{\overline{\star}}$, then necessarily $G^\star = (D : F)^\star$, since $\overline{\star} \leq \star$.

Let F^{-1} be $\star_f$–finite. If $G \in \boldsymbol{f}(D)$ is such that $G \subseteq (D : F)$ and $G^\star = (D : F)^\star$. Since F is $\star$–invertible then $(FG)^\star = (F(D : F))^\star = D^\star$, thus $FG \in \mathcal{F}^\star$. Since $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}}$, then $(FG)^{\overline{\star}} = D^{\overline{\star}} = (F(D : F))^{\overline{\star}}$. Therefore $((FG)^{\overline{\star}}(D : F)^{\overline{\star}})^{\overline{\star}} = (D^{\overline{\star}}(D : F)^{\overline{\star}})^{\overline{\star}}$, i.e. $G^{\overline{\star}} = (D : F)^{\overline{\star}}$.

Last statement is a straightforward consequence of (ii)$\Rightarrow$(i) and of the fact that in a $\star_f$–quasi-coherent, for each $F \in \boldsymbol{f}(D)$, F^{-1} is $\star_f$–finite. $\square$

Remark 11. (1) Note that last statement of Proposition 12 may not be improved by replacing $\mathtt{I}(\star)$–domain with $\mathtt{H}(\star)$–domain: Example 6 provides *a t–quasi-coherent domain* (since PvMD, Theorem 2 ((i)$\Rightarrow$(v$_f$))) *which is not an $\mathtt{H}$–domain.*

(2) The following Remark 13 (2) shows that the converse of the last statement of Proposition 12 does not hold: it is easy to see that there exists *an example of an* $\mathtt{I}(\star)$*–domain which is not a* $\star_{_f}$*–quasi-coherent domain.*

The following result improves Corollary 6 (cf. also Theorem 2 ((i)$\Leftrightarrow$(v$_f$)) and last statement of Proposition 12).

Corollary 7. *Let D be an integral domain and $\star$ a semistar operation on D. The following conditions are equivalent:*

(i) *D is a $\star$-domain and an $\mathtt{I}(\star)$–domain.*
(ii) *D is a P$\star$MD.*
(iii) *D is a P$\overline{\star}$MD.*

Proof. (i)$\Leftrightarrow$(ii) Recall that, if $F \in \boldsymbol{f}(D)$, I is $\star_{_f}$–invertible if and only if I is $\star$–invertible and F^{-1} is $\star_{_f}$–finite [17, Proposition 2.6]. Therefore this equivalence follows easily from Proposition 12 ((ii)$\Leftrightarrow$(i)) and Proposition 2 (3).

(ii)$\Leftrightarrow$(iii) By Proposition 12 ((i)$\Leftrightarrow$(iii)) and Proposition 2 (4), this equivalence is a straightforward consequence of (ii)$\Leftrightarrow$(i). $\square$

Remark 12. Note that Example 6 provides also *an example of an* $\mathtt{I}(\star)$*–domain which is not an* $\mathtt{H}(\star)$*–domain,* since, by Corollary 7, a PvMD is also a $\mathtt{I}(v)$–domain.

The considerations in Example 6 and the equivalence (i)$\Leftrightarrow$(iii$_f$) in Theorem 2 lead us also to consider another weaker condition of the property of being an $\mathtt{H}(\star)$–domain (i.e. $(\widetilde{\star}, \overline{\star})$–domain), namely the notion of $(\widetilde{\star}, (\overline{\star})_{_f})$–domain. It is obvious (like for the $\mathtt{H}(\star)$–domain case) that when $\star = \star_{_f}$ an integral domain is trivially a $(\widetilde{\star}, (\overline{\star})_{_f})$–domain.

Recall that, it was shown in [11] that $\widetilde{\star} \leq (\overline{\star})_{_f}$ but in general these two semistar operations do not coincide [11, Example 3.11].

Proposition 13. *Let D be an integral domain and $\star$ a semistar operation on D. Assume that D is either an $\mathtt{H}(\star)$–domain or truly $\star_{_f}$-coherent. Then D is a $(\widetilde{\star}, (\overline{\star})_{_f})$–domain.*

Proof. The case of an $\mathtt{H}(\star)$–domain is immediate since it is a $(\widetilde{\star}, \overline{\star})$–domain (Proposition 11 ((i)$\Rightarrow$(vii))).

So, assume that D is truly $\star_{_f}$–coherent and let I be a nonzero finitely generated ideal of D. We have only to show that $I^{\overline{\star}} \subseteq I^{\widetilde{\star}}$. Let $x \in I^{\overline{\star}}$. Then, there exists $J \in \mathcal{F}^{\star}$ such that $xJ \subseteq I$. Since $J \subseteq x^{-1}I \cap D$, we have that $x^{-1}I \cap D \in \mathcal{F}^{\star}$. Moreover, $x^{-1}I, D \in \boldsymbol{f}(D)$, so by the assumption $x^{-1}I \cap D$ is $\star_{_f}$–finite. Let $H \subseteq x^{-1}I \cap D$ be a nonzero finitely generated ideal such that $H^{\star_f} = H^{\star} = (x^{-1}I \cap D)^{\star} = D^{\star}$. It follows that $H \in \mathcal{F}^{\star_f}$. Moreover $xH \subseteq x(x^{-1}I \cap D) \subseteq I$, and so $x \in I^{\widetilde{\star}}$. Hence $(\overline{\star})_{_f} = \widetilde{\star}$. $\square$

Remark 13. **(1)** Note that Example 6 shows that *a $\widetilde{\star}$–extracoherent domain (or, a truly $\star_{_f}$–coherent domain) which is also a $\star$–domain is not necessarily an* $\mathtt{H}(\star)$*–domain.*

(2) Note that, if $\star = \star_{_f}$, then properties of being an $\mathtt{H}(\star)$–domain, an $\mathtt{I}(\star)$–domain and a $(\widetilde{\star}, (\overline{\star})_{_f})$–domain are all trivially satisfied (recall that $\overline{(\star_{_f})} = \widetilde{\star}$ by Proposition 1 (7)). So, in particular, none of them implies the $\star$–quasi-coherence and the $\star$–coherence (and so neither the $\star$–ultracoherence nor the truly $\star$–coherence). Indeed, it is enough to take $\star = d$ and consider an arbitrary non-(d–)quasi-coherent domain (e.g. a non-finite conductor domain [21]). This example shows, in particular, that there exists *an example of an* $\mathtt{I}(\star)$*–domain which is not a* $\star_{_f}$*–quasi-coherent domain (cf. also the following question* **(Q-3)** in Remark 14 (1)).

Recall that we already know that:

$$\begin{array}{rcl} \text{truly } \star_{_f}\text{–coherent domain} & \Rightarrow & (\widetilde{\star}, (\overline{\star})_{_f})\text{–domain (Proposition 13)}, \\ \star_{_f}\text{–quasi-coherent domain} & \Rightarrow & \mathtt{I}(\star)\text{–domain (Proposition 12)}, \quad \text{and} \\ \text{truly } \star_{_f}\text{–coherent domain} & \Rightarrow & \star_{_f}\text{–quasi-coherent domain (Theorem 1 (2))}. \end{array}$$

The next goal is to relate the notions of $(\widetilde{\star}, (\overline{\star})_{_f})$–domain and $\mathtt{I}(\star)$–domain.

Proposition 14. *Let D be an integral domain and $\star$ a semistar operation on D. If D is a $(\widetilde{\star}, (\overline{\star})_{_f})$–domain then D is an $\mathtt{I}(\star)$–domain.*

Proof. Assume that I is a finitely generated $\star$–invertible ideal. We already remarked that this is equivalent to the fact that I is $\overline{\star}$–invertible. We want to show that I is $\widetilde{\star}$–invertible (and so, $\star_{_f}$–invertible [17, Proposition 2.18]). By [16, Proposition 5.3 (2)], it is enough to show that I is $\widetilde{\star}$–e.a.b., i.e., that if $(IF)^{\widetilde{\star}} = (IG)^{\widetilde{\star}}$ for some $F, G \in \boldsymbol{f}(D)$ then $F^{\widetilde{\star}} = G^{\widetilde{\star}}$.

Since, by assumption, $\overline{\star}$ and $\widetilde{\star}$ coincide on $\boldsymbol{f}(D)$, we have $(IF)^{\overline{\star}} = (IG)^{\overline{\star}}$ and so $F^{\overline{\star}} = G^{\overline{\star}}$, because I is $\overline{\star}$-invertible. Thus, since $F, G \in \boldsymbol{f}(D)$, again by the assumption, we have $F^{\widetilde{\star}} = G^{\widetilde{\star}}$. □

Corollary 8. *Let D be an integral domain and $\star$ a semistar operation on D. Assume that D is a $\star$–domain. The following are equivalent:*

- **($\mathbf{i}_{\boldsymbol{f}}$)** *$D$ is a truly $\star_{_f}$–coherent domain.*
- **($\mathbf{ii}_{\boldsymbol{f}}$)** *$D$ is a $\star_{_f}$–quasi-coherent domain.*
- **(iii)** *D is a $(\widetilde{\star}, (\overline{\star})_{_f})$–domain.*
- **(iv)** *D is an $\mathtt{I}(\star)$–domain.*

Proof. We already observed that in general (i$_f$)⇒(ii$_f$), (iii)⇒(iv), (i$_f$)⇒(iii) and (ii$_f$)⇒(iv) (Theorem 1 (2), Proposition 12, Proposition 13 and Proposition 14).

(iv)⇒(i$_f$) When D is a $\star$–domain, an $\mathtt{I}(\star)$–domain is a P$\star$MD and thus (i$_f$) holds (Corollary 7 ((i)⇒(ii)) and Theorem 2 ((i)⇒(iii$_f$)). □

Example 7. *An $\mathtt{I}(\star)$–domain D which is not a P$\star$MD (and so, in particular, which is not a $\star$–domain).*

Take any integral domain domain which is not Prüfer and take $\star = d$.

Remark 14. **(1)** The characterizations in Corollary 7 lead to investigate more in depth the class of $\mathtt{I}(\star)$–domains. In particular, in relation with Proposition 12, Proposition 14 and Corollary 8, it is natural to consider the following question (with $\star \neq d$ in order to avoid the trivial cases):

(Q-3) *Is it possible to find an example of an $\mathtt{I}(\star)$–domain which is not a $(\widetilde{\star}, (\overline{\star})_{_f})$–domain or which is not a $\star_{_f}$–quasi-coherent domain (and so, also, not a $\star$–domain) ? Is it possible to find such examples with $\star = v$?*

(2) Since the notions of $\star$–domain and $\overline{\star}$–domain [respectively: the notions of $\mathtt{I}(\star)$–domain and $\mathtt{I}(\overline{\star})$–domain; the notions of P$\star$MD and P$\overline{\star}$MD] coincide (Proposition 2 (4) [respectively: Proposition 12; Remark 1 (4)]), we have that, *if D is a $\star$–domain, the conditions (i)–(iv) of Corollary 8 are also equivalent to each of the following:*

- **($\overline{\mathbf{i}}_{\boldsymbol{f}}$)** *$D$ is a truly $(\overline{\star})_{_f}$–coherent domain.*
- **($\overline{\mathbf{ii}}_{\boldsymbol{f}}$)** *$D$ is a $(\overline{\star})_{_f}$–quasi-coherent domain.*

and, using also Theorem 2, *these conditions are also equivalent to each of the following:*

- **($\widetilde{\mathbf{i}}$)** *D is a (truly) $\widetilde{\star}$–coherent domain.*
- **($\widetilde{\mathbf{ii}}$)** *D is a $\widetilde{\star}$–quasi-coherent domain.*

and also to each of the following:

- **(j)** *D is a $\star$–extracoherent (or, equivalently, $\star_f$–extracoherent) domain.*
- **($\overline{\text{j}}$)** *D is a $\overline{\star}$–extracoherent (or, equivalently, $(\overline{\star})_f$–extracoherent) domain.*
- **($\widetilde{\text{j}}$)** *D is a $\widetilde{\star}$–extracoherent domain.*

The equivalence (ii)$\Leftrightarrow$(iv) in the following theorem provides evidence for question **(Q-1)** in Remark 3 (4).

Theorem 3. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

- **(i)** *D is a $\star$–domain and a $(\widetilde{\star}, \star_f)$–domain.*
- **($\overline{\text{i}}$)** *D is a $\overline{\star}$–domain and a $(\widetilde{\star}, (\overline{\star})_f)$–domain.*
- **(ii)** *$\star$ is a.b. and D is a $(\widetilde{\star}, \star_f)$–domain.*
- **($\overline{\text{ii}}$)** *$\overline{\star}$ is a.b. and D is a $(\widetilde{\star}, (\overline{\star})_f)$–domain.*
- **(iii)** *$\star$ is e.a.b. and D is a $(\widetilde{\star}, \star_f)$–domain.*
- **($\overline{\text{iii}}$)** *$\overline{\star}$ is e.a.b. and D is a $(\widetilde{\star}, (\overline{\star})_f)$–domain.*
- **(iv)** *D is a P$\star$MD.*

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii), ($\overline{\text{i}}$)$\Rightarrow$($\overline{\text{ii}}$)$\Rightarrow$($\overline{\text{iii}}$) and (i)$\Rightarrow$($\overline{\text{i}}$) follow from Proposition 3, Proposition 2 (4) and from the fact that $\widetilde{\star} \leq (\overline{\star})_f \leq \star_f$ [11, Proposition 3.6].

(iii)$\Rightarrow$(iv) and ($\overline{\text{iii}}$)$\Rightarrow$(iv) Since $\star$ [respectively: $\overline{\star}$] is e.a.b. then $\star_f = \widetilde{\star}$ [respectively: $(\overline{\star})_f = \widetilde{\star}$] is (e.)a.b. [16, Lemma 3.8] and so D is a P$\star$MD by [13, Theorem 3.1 ((v))$\Rightarrow$(i))].

(iv)$\Rightarrow$(i) Recall that a P$\star$MD is an integral domain such that $\star_f = \overline{(\star_f)} = \widetilde{\star}$ and is (e.)a.b. [13, Theorem 3.1 ((i)$\Leftrightarrow$(vi))]. Moreover we know that a P$\star$MD is a $\star$–domain (Proposition 2 (3)). $\square$

Remark 15. (1) Note that Example 6 provides *an example of a $(\widetilde{\star}, \star_f)$–domain (so, in particular, a $(\widetilde{\star}, (\overline{\star})_f)$–domain, hence also an $\mathtt{I}(\star)$–domain, Proposition 14) which is not a $(\widetilde{\star}, \overline{\star})$–domain (i.e. an $\mathtt{H}(\star)$–domain),* since a P$\star$MD is a $(\widetilde{\star}, \star_f)$–domain (Theorem 3 ((i)$\Leftrightarrow$(iv)).

(2) Example 2 provides also *an example of an integral domain D and a semistar operation $\star$ on D such that D is a $\star$–domain but D does not verify any of the (equivalent) conditions of Corollary 8 and Remark 14 (2).*

(3) Note that the implication (i)$\Rightarrow$(iv) in Theorem 3 could be also proved directly by applying Proposition 14 and Corollary 7 ((i)$\Rightarrow$(ii)).

In Theorem 3, after proving that (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv), then the equivalences ($\overline{\text{i}}$)$\Leftrightarrow$($\overline{\text{ii}}$)$\Leftrightarrow$($\overline{\text{iii}}$)$\Leftrightarrow$(iv) could be also obtained from Corollary 7 ((ii)$\Leftrightarrow$(iii)) and from the fact that $\widetilde{\overline{\star}} = \widetilde{\star}$ [13, Corollary 3.9].

(4) It is easy to see that *the statements in Theorem 3 are also equivalent to the following:*

- **(i′)** *D is a $\overline{\star}$–domain and a $(\widetilde{\star}, \star_f)$–domain.*
- **(i″)** *D is a $\star$–domain and a $(\widetilde{\star}, (\overline{\star})_f)$–domain.*
- **(ii′)** *$\overline{\star}$ is a.b. and D is a $(\widetilde{\star}, \star_f)$–domain.*
- **(iii′)** *$\overline{\star}$ is e.a.b. and D is a $(\widetilde{\star}, \star_f)$–domain.*

But, in general, they are not equivalent to the following (weaker) statements:

- **(ii″)** *$\star$ is a.b. and D is a $(\widetilde{\star}, (\overline{\star})_f)$–domain.*
- **(iii″)** *$\star$ is e.a.b. and D is a $(\widetilde{\star}, (\overline{\star})_f)$–domain.*

As a matter of fact, if $\star = \star_f$, the condition "$(\widetilde{\star}, (\overline{\star})_f)$–domain" is trivially satisfied, but the condition "$\star$ is a.b." (or "$\star$ is e.a.b.") does not imply P$\star$MD for lack of stability.

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